Arithmeticity of L-functions for Quaternionic Groups

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Deligne's Conjecture on Critical Values of L-functions

- Motivation: Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function. For positive even integers k,

$$\zeta(k) = (-1)^{\frac{k}{2}+1} \frac{(2\pi)^k B_k}{2(k!)}.$$

- General conjecture (Deligne): if L is a motivic L-function, then

$$L(k) \in (\text{period}) \cdot \overline{\mathbb{Q}}$$

  at "critical values".
- One method to prove things about (automorphic) L-functions is to use integral representations and properties of Eisenstein series.

A result of Shimura

- Let f and g be holomorphic modular forms with Fourier expansions

$$f = \sum_{n=0}^{\infty} a_n e^{2\pi inz}, g = \sum_{n=0}^{\infty} b_n e^{2\pi inz}$$

- Define the product L-function

$$L(s, f \times g) = \sum_{n=0}^{\infty} a_n \overline{b_n} n^{-s}$$

Theorem (Shimura)

Let f be a Hecke eigenform of weight ℓ_1 and g a holomorphic modular form of weight $\ell_2 < \ell_1$. Then, when k is an integer with $\frac{1}{2}(\ell_1 + \ell_2 - 2) < s < \ell_1$,

$$\pi^{-\ell_1} \frac{L(k, f \times g)}{\langle f, f \rangle} \in \mathbb{Q}(f)\mathbb{Q}(g)$$

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- Proof of theorem: Integral representation, control of Fourier coefficients and properties of Eisenstein series, Maass-Shimura operators
- Integral representation (Rankin, Selberg):

  $$\langle f(z), g(z) \cdot E_n(z, s) \rangle \approx L(s + \ell_1 - 1, f \times g),$$

  where $E_n(z, s)$ is real-analytic Eisenstein series of weight $n = \ell_1 - \ell_2$.
- When $s = 0$, $E_n(z, 0)$ is a holomorphic Eisenstein series of weight n. So

  $$\langle f, g \cdot E_n(z, 0) \rangle \approx L(\ell_1 - 1, f \times g),$$

  which implies

  $$\pi^{-\ell_1} \langle f, f \rangle^{-1} L(\ell_1 - 1, f \times g) \in \mathbb{Q}(f)\mathbb{Q}(g).$$

A result of Shimura

- We have the result for the *right-most* critical value in Shimura's theorem.
- To get algebraicity results for critical values to the left of $\ell_1 - 1$, use Maass-Shimura differential operators

$$\delta_n = \frac{1}{2\pi i}\left(\frac{n}{2iy} + \frac{\partial}{\partial z}\right), \delta_n^{(r)} = \delta_{n+2r-2} \circ \cdots \circ \delta_{n+2} \circ \delta_n$$

- Then $E_{n+2r}(z, -r) \approx \delta_n^{(r)} E_n(z, 0)$ and

$$\langle f, g \cdot \delta_n^{(r)} E_n(z, 0)\rangle \approx \langle f, g \cdot E_{k-n}(z, -r)\rangle \approx L(\ell_1 - 1 - r, f \times g).$$

- Conclusion: algebraicity of $\pi^{-\ell_1}\langle f, f\rangle^{-1} L(\ell_1 - 1 - r, f \times g)$
- Inner workings: $G = g \cdot \delta_n^{(r)} E_n(z, 0)$ is not a modular form, but integrating against it is the same as integrating against a modular form G_0. The Fourier coefficents of G_0 lie in $\mathbb{Q}(g)\mathbb{Q}(E_n(z, 0))$. In fact G_0 is proportional to the *Rankin-Cohen bracket* $[g, E_n]_r$.

More on Shimura's proof

- There is a representation theoretic perspective (Harris).
- Let π_m be the holomorphic discrete series representation of SL_2 with lowest K-type $\mathbb{C}(m)$. Then (Vergne),

$$\pi_m \otimes \pi_n = \bigoplus_{j=0}^{\infty} \pi_{m+n+2j}.$$

- If g is weight m, then it corresponds to a vector v_m in the lowest K-type of π_m; similarly for E and $v_n \in \pi_n$.
- The Maass-Shimura operator is $X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathfrak{sl}_{2,\mathbb{C}}$.
- G_0 is the projection of $v_m \otimes X^r v_n$ to π_{m+n+2r}.
- In this case, one can calculate an explicit formula for G_0.

Outline for remainder of talk

- Shimura's method has been expanded and generalized to many higher rank situations, e.g. standard L-function of Siegel modular forms (Harris, Horinaga-Pitale-Saha-Schmidt) and spin L-function of GSp_6 (Eischen-Rosso-Shah).
- We will describe an application of the technique to a non-holomorphic setting, quaternionic modular forms (Gan-Gross-Savin, Pollack).
- There is a class of groups that, unlike SL_2 or Sp_{2n}, don't necessarily have holomorphic discrete series representations. However they have *quaternionic discrete series* (Gross-Wallach).
- Examples: $\mathrm{SU}(2,n), G_2, \mathrm{Spin}(4,4), F_4, E_6, E_7, E_8$.
- We will:
  1. Describe an integral representation of an L-function that is amenable to quaternionic data;
  2. Review the ingredients necessary to prove algebraicity results;
  3. Focus on describing an analog of Maass-Shimura operators for quaternionic modular forms;
  4. If there is time at the end, more about the arithmeticity of quaternionic Eisenstein series.

Outline for remainder of talk

- Hundley found an integral representation for the adjoint L-function of $\mathrm{SU}(2,1)$. It relies on an embedding $\mathrm{SU}(2,1) \hookrightarrow G_2$.
- Let Π be a cuspidal automorphic representation of $\mathrm{SU}(2,1)$, quaternionic of weight ℓ at infinity. Let $\varphi \in \Pi$.
- Let $E_\ell(g,s)$ be a certain degenerate Eisenstein series on G_2. Then,

$$\langle \varphi, E_\ell(g,s) \rangle \approx L(s-1, \Pi, \mathrm{Ad}).$$

- If $s = \ell + 1$, then $E_\ell(g, s = \ell + 1)$ is a quaternionic modular form. At the same time, ℓ is the right-most critical value of $L(s, \Pi, \mathrm{Ad})$.

Algebraicity Results

- Ingredient 1, Integral representation: ✓
- Ingredient 2, Control of Fourier coefficients / properties of Eisenstein series: When $s = \ell + 1$, then $E_\ell(g, s = \ell + 1)$ is a QMF.

Theorem (ongoing joint work with J. Johnson-Leung, F. McGlade, A. Pollack, M. Roy)

The degenerate quaternionic Heisenberg Eisenstein series on G_2 (and $B_3, D_4, F_4, E_6, E_7, E_8$) can be normalized to have algebraic Fourier coefficients.

- Cook these up: taking an eigenform $\varphi \in \Pi$

$$\frac{L(\ell, \Pi, \mathrm{Ad})}{\langle \varphi, \varphi \rangle} \in \pi^{\mathbb{Z}} \cdot \mathbb{Q}(\varphi).$$

- To get algebraicity results for critical values to the left, we need Ingredient 3, Differential Operators.

Exceptional Maass-Shimura Operators

Theorem (H.)

Let F be a QMF on $G = G_2$ of weight n. For any integers $r \geqslant 0$ and $m \in [r] = \{-r, -r+2, \ldots, r-2, r\}$, there is a differential operator $\mathcal{D}^n_{r,m}$ such that:

- $f = (\mathcal{D}^n_{r,m} F)|_H$ *is a QMF on* $H = \mathrm{SU}(2,1)$ *of weight* $(n + \frac{r}{2}, m)$.
- *The Fourier coefficients of f are $\mathbb{Q}(i)$-linear combinations of the Fourier coefficients of F.*

- We will discuss how to find explicit recurrence formulas for the "highest-weight" part of $\mathcal{D}^n_{r,m}$.
- We need these formulas to prove the relationship between Fourier coefficients.
- Applying these operators to quaternionic Eisenstein series on $G = G_2$ allows us to access the critical values of $L(s, \Pi, \mathrm{Ad})$.
- For the application, we only really need $\mathcal{D}^n_{2r,0}$. However everything is proved by induction and the stronger statement are necessary.

Group Theory

- The *split octonions* Θ over $\mathbb{Q}$, for example as defined by the Cayley-Dickson construction:
$$\Theta = \{(x, y) : x, y \in B = M_2(\mathbb{Q})\}.$$
- Let $G = G_2$ be the automorphism group of Θ.
- If $v \in \Theta$ has squarefree norm $D > 0$, then $H = \mathrm{Stab}_G(v)$ is a subgroup of type $\mathrm{SU}(2, 1)$.
- In more detail, $\mathbb{Q}(v) \cong \mathbb{Q}(\sqrt{-D})$ and the orthogonal complement of $\mathbb{Q}(v)$ in Θ is a 3-dimensional $\mathbb{Q}(v)$ Hermitian space.
- For convenience, let $v = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right)$ so that $\mathbb{Q}(v) = \mathbb{Q}(i)$ and $H \hookrightarrow G$ is in "good position".

More group theory

- The maximal compact subgroups:

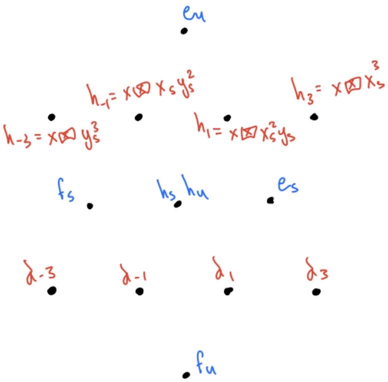
$$K_H := \mathrm{SU}(2)^{\mathrm{long}} \times U(1)/\mu_2 \subset \mathrm{SU}(2)^{\mathrm{long}} \times \mathrm{SU}(2)^{\mathrm{short}}/\mu_2 =: K_G$$

- The Cartan decomposition: Let $\mathfrak{g}$ be the complexified Lie algebra of $G = G_2$. Then

$$\mathfrak{g} = (\mathfrak{sl}_2^{\mathrm{long}} \oplus \mathfrak{sl}_2^{\mathrm{short}}) \oplus \mathfrak{p}$$

  where $\mathfrak{p} \cong V_2^{\mathrm{long}} \oplus \mathrm{Sym}^3(V_2^{\mathrm{short}})$ as a representation of K_G.

Picture and notation



Long root $\mathfrak{sl}_2$: $\{e_u, h_u, f_u\}$

Short root $\mathfrak{sl}_2$: $\{e_s, h_s, f_s\}$

$$\mathfrak{p} = \mathrm{Span}_{\mathbb{C}}\{h_{\pm j}, \delta_{\pm j}\}$$

$$\mathrm{Lie}(H) \otimes \mathbb{C} = \mathrm{Span}_{\mathbb{C}}\left\{\begin{matrix} e_u, h_u, f_u, h_r, \\ h_{\pm 3}, \delta_{\pm 3} \end{matrix}\right\}$$

Quaternionic Modular Forms

- Let $n \geqslant 1$. There is a (limit of) discrete series representation π_n^G of $G = G_2$, with lowest K_G-type $\mathbb{V}_n = \mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathbf{1}$.

Definition (Gan-Gross-Savin, A. Pollack)

A quaternionic modular form (QMF) on $G = G_2$ of weight n is a smooth function $F : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{V}_n^\vee$ satisfying:

1. $F(\gamma g) = \Phi(g)$ for all $\gamma \in G_2(\mathbb{Q})$ and $g \in G_2(\mathbb{A})$
2. $F(gk) = k^{-1}\Phi(g)$ for all $k \in K_G$ and $g \in G_2(\mathbb{A})$
3. $D_n F = 0$ for a certain differential operator D_n

- One can analogously make a definition for QMFs on $H = \mathrm{SU}(2,1)$. These are associated to quaternionic (i.e. “large” or “generic”) nonholomorphic discrete series representations $\pi^H_{n+\frac{r}{2},m}$ of H, with lowest K-type $\mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes \mathbb{C}(m)$.

On Discrete Series

The representation π_n^G has K_G-type decomposition

$$\pi_n^G = \bigoplus_{r=0}^{\infty} \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}})).$$

The representation $\pi^H_{n+\frac{r}{2},m}$ has K_H-type decomposition

$$\pi^H_{n+\frac{r}{2},m} = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n+r+j}(V_2^{long}) \boxtimes (\mathrm{Sym}^j(\mathbb{C}(-1) \oplus \mathbb{C}(1)) \otimes \mathbb{C}(m)).$$

Theorem (H. Y. Loke)

For a nonnegative integer r, let $[r] := \{-r, -r+2, \ldots, r-2, r\}$. Then, as representations of H,

$$\pi_n^G|_H = \bigoplus_{r=0}^{\infty} \bigoplus_{m \in [r]} \pi^H_{n+\frac{r}{2},m}.$$

On Discrete series

The representation π_n^G has K_G-type decomposition

$$\pi_n^G = \bigoplus_{r=0}^{\infty} \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}})).$$

- By Loke's restriction theorem, there exists a unique line $L_{r,m}^n$ in $\mathrm{Sym}^r(V_G)$ so that
$$\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L_{r,m}^n \subseteq \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)$$
is the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m}$ in $\pi_n^G|_H$.
- This tells us that $\mathcal{D}_{r,m}^n$ is
$$\mathrm{Proj}_{\mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}})\boxtimes L_{r,m}^n} \circ \widetilde{D}^r,$$
where
$$\widetilde{D}^r F = \sum_i X_i F \otimes X_i^{\vee}.$$
- In order to prove any relationship between Fourier coefficients, we want to make this effective, i.e. pin down $L_{r,m}^n$ explicitly.

On Discrete Series

The representation π_n^G has K_G-type decomposition

$$\pi_n^G = \bigoplus_{r=0}^{\infty} \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}})).$$

- Goal: Find explicit formulas for elements $D_{r,m}^n \in U(\mathfrak{p})$ that take x^{2n} in the lowest K_G-type $\mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathbf{1}$ of π_n^G to to x^{2n+r} in the lowest K_H type of $\pi^H_{n+\frac{r}{2},m}$.
- Let $\ell_{r,m}^n$ be a (suitably normalized) basis element for $L_{r,m}^n$. We can prove that

$$\langle \mathcal{D}_{r,m}^n F, x^{2n+r} \boxtimes \ell_{r,m}^n \rangle = D_{r,m}^n \langle F, x^{2n} \rangle.$$

- Recall that $\mathfrak{p} \cong V_2^{long} \boxtimes \mathrm{Sym}^3(V_2^{\mathrm{short}})$.
- To move between K_G-types: use elements of $\mathfrak{p}^r \subseteq U(\mathfrak{g})$.
- More precisely, we find elements $D_{r,m}^n \in \mathbb{C}[h_{-3}, h_{-1}, h_1, h_3]$ (whose action on x^{2n} is well-defined, e.g. by PBW theorem).

The Highest Weight Operators

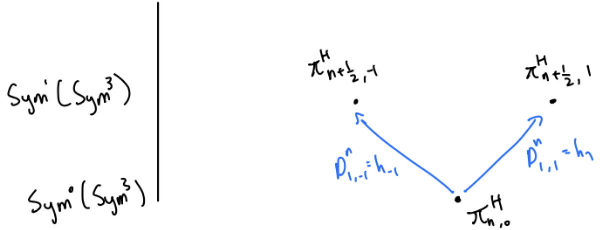
We are starting with a fix weight $n \geqslant 1$. Define, for $r \geqslant 0$ and $m \in [r] := \{-r, -r+2, \dots, r-2, r\}$,

- $A^n_{r,m} := -\frac{1}{3}\frac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
- $B^n_{r,m} := -\frac{1}{9}\frac{(4n+r+m-2)(3n+r-2)(n+r-1)(r+m)}{(2n+r+m)(2n+r+m-2)(2n+r-1)(2n+r-2)}$
- $E^n_{r,r} := A^n_{r,-r}$

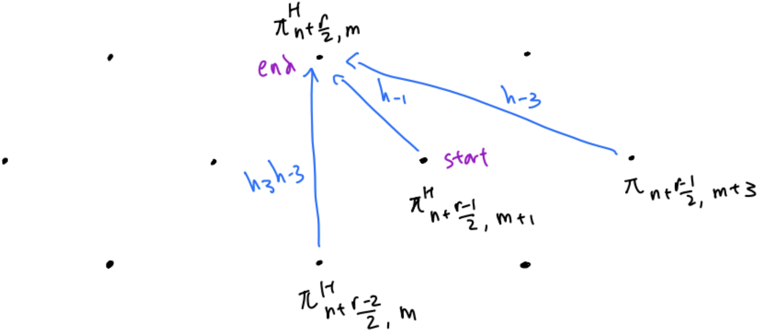
Definition

Define recursively $D^n_{0,0} = 1, D^n_{1,1} = h_1, D^n_{1,-1} = h_{-1}$, and

- $D^n_{r,m} = h_{-1}D^n_{r-1,m+1} + A^n_{r,m}h_{-3}D_{r-1,m+3} + B^n_{r,m}h_3h_{-3}D^n_{r-2,m}$ for $m < r$.
- $D^n_{r,r} = h_1D^n_{r-1,m-1} + E^n_{r,r}h_3D_{r-1,m-3}$



- Recall that
$$\pi_n^G = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathrm{Sym}^j(\mathrm{Sym}^3(V_2^{\mathrm{short}})).$$
- The first row depicts $j = 0$, i.e. $\mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathbf{1}$
- The second row depics $j = 1$, i.e. $\mathrm{Sym}^{2n+1}(V_2^{\mathrm{long}}) \boxtimes \mathrm{Sym}^3(V_2^{\mathrm{short}})$. It looks like there are parts missing; these are the things that come from higher K_H-types of $\pi^H_{n,0}$.



$$D^n_{r,m} = h_{-1}D^n_{r-1,m+1} + A^n_{r,m}h_{-3}D_{r-1,m+3} + B^n_{r,m}h_3h_{-3}D^n_{r-2,m}$$

The Highest Weight Operators

- $A^n_{r,m} := -\dfrac{1}{3}\dfrac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
- $B^n_{r,m} := -\dfrac{1}{9}\dfrac{(4n+r+m-2)(3n+r-2)(n+r-1)(r+m)}{(2n+r+m)(2n+r+m-2)(2n+r-1)(2n+r-2)}$
- $E^n_{r,r} := A^n_{r,-r}$

Theorem (H.)

Let x^{2n} be a highest-weight vector in the lowest K_G-type of π^G_n. The vector $D^n_{r,m}x^{2n}$ is a highest-weight vector in the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m} \subseteq \pi^G_n|_H$.

- In order to motivate the proof, and the formulas for $D^n_{r,m}$, we describe an algorithm to explicitly compute $D^n_{r,m}$ for any n, r, m.
- Key: the Casimir element Ω_H acts on $\pi^H_{n+\frac{r}{2},m}$ as a scalar $\lambda^n_{r,m}$.

Algorithm

- Start with $v = x^{2n}$ in the lowest K_G-type.
- $D^n_{r,m}$ is a linear combination of $h^a_{-3}h^b_{-1}h^c_1h^d_3$ with $a+b+c+d=r$ and $-3a-b+c+3d=m$. Let
$$X = \sum R_{a,b,c,d} \cdot h^a_{-3}h^b_{-1}h^c_1h^d_3$$
be a general linear combination of such elements.
- Calculate
$$(\Omega_H X - X\Omega_H)v = \left(\sum S_{a,b,c,d} \cdot h^a_{-3}h^b_{-1}h^c_1h^d_3\right) v,$$
where each $S_{a,b,c,d}$ is a $\mathbb{Q}$-linear combination of all the R's.
- Linear algebra problem: Find $R_{a,b,c,d}$ so that
$$(\Omega_H X - X\Omega_H)v = (\lambda^n_{r,m} - \lambda^n_{0,0})Xv.$$
- Then $\Omega_H X v = \lambda^n_{r,m} X v$ and we can set $D^n_{r,m}$ to (some normalization of) X. Done!

Remark

This computation boils down to the commutators $[\Omega_H, h^b_{-1}h^c_1]$. Inspecting these is how one can "guess" the shape of the recurrence formulas for $D^n_{r,m}$.

The real theorem statement

- To prove that

$$D^n_{r,m}v = (h_{-1}D^n_{r-1,m+1} + A^n_{r,m}h_{-3}D^n_{r-1,m+3} + B^n_{r,m}h_3h_{-3}D^n_{r-2,m})v$$

  is in the correct piece of $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_2^{short})$, one computes the commutator

$$[\Omega_H, h_{-1}] = h_{-1}\left(\frac{2}{3} - \frac{1}{3}h_s + h_u\right) + 2d_{-1}e_u - \frac{2}{3}h_{-3}e_s.$$

- The only problematic term is e_s. Since $e_s \cdot v = 0$, we need to understand $[e_s, D^n_{r,m}]$.
- In the same vein, we need to understand $[f_s, D^n_{r,m}]$.

The real theorem statement

- $A^n_{r,m} := -\dfrac{1}{3}\dfrac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
- $B^n_{r,m} := -\dfrac{1}{9}\dfrac{(4n+r+m-2)(3n+r-2)(n+r-1)(r+m)}{(2n+r+m)(2n+r+m-2)(2n+r-1)(2n+r-2)}$
- $E^n_{r,r} := A^n_{r,-r}$

Theorem (H.)

Let x^{2n} be a highest-weight vector in the lowest K_G-type of π^G_n. The vector $D^n_{r,m}x^{2n}$ is a highest-weight vector in the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m} \subseteq \pi^G_n|_H$.

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- $E^n_{r,m} := A^n_{r,-m}$
- $F^n_{r,m} := B^n_{r,-m}$
- $U^n_{r,m} := \dfrac{1}{2}\dfrac{(4n+r+m-2)(r+m)}{(2n+r+m-2)}$
- $V^n_{r,m} := \dfrac{1}{3}\dfrac{(4n+r-m-2)(3n+r-1)(n+r)(r-m)}{(2n+r-m)(2n+r-m-2)(2n+r-1)}$
- $S^n_{r,m} := U^n_{r,-m}$
- $T^n_{r,m} := V^n_{r,-m}$

The real theorem statement

Theorem (H.)

Let $n \geqslant 1$. Recall the (limit of) discrete series representation π_n^G has lowest K_G-type $\mathbb{V}_n = \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. Let $v = x^{2n} \in \mathbb{V}_n$ be a highest weight vector. For any $r \geqslant 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,

1. *$D^n_{r,m} v$ is an Ω_H-eigenvector with eigenvalue $\lambda^n_{r,m}$, and $D^n_{r,m} v \in x^{2n+r} \boxtimes L^n_{r,m}$ where $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^n_{r,m} \subseteq \pi_n^G$ is the lowest K_H-type of the unique $\pi^H_{n+\frac{r}{2},m} \subseteq \pi_n^G|_H$.*
2. $D^n_{r,m} = h_1 D^n_{r-1,m-1} + E^n_{r,m} h_3 D^n_{r-1,m-3} + F^n_{r,m} h_3 h_{-3} D^n_{r-2,m}$
3. $[f_s, D^n_{r,m}] = U^n_{r,m} D^n_{r,m-2} + V^n_{r,m} h_{-3} D^n_{r-1,m+1}$
4. $[e_s, D^n_{r,m}] = S^n_{r,m} D^n_{r,m+2} + T^n_{r,m} h_3 D^n_{r-1,m-1}$

Details on the Fourier expansion of QMFs

- $G = G_2$ has two (conjugacy classes of) maximal parabolic subgroups.
- The Heisenberg parabolic has Levi decomposition $P = MN$ with $M \cong GL_2$, $Z = [N, N]$, and N/Z isomorphic to the space of binary cubic forms.
- We will write $\omega = (a, b/3, c/3, d)$ for the binary cubic form $au^3 + bu^2v + cuv^2 + dv^3$.
- The Fourier expansion (along the center of the Heisenberg parabolic) of a QMF on $G = G_2$ is indexed by positive semi-definite binary cubic forms: For $g = g_f g_\infty \in G(\mathbb{A}_f)G(\mathbb{R})$,

$$F_Z(g) = F_N(g) + \sum_{\omega \geqslant 0} a_{F,\omega}(g_f) W_{2\pi\omega,n}(g_\infty)$$

  where $a_{F,\omega}$ is a locally constant Fourier coefficient and $W_{2\pi\omega,n}$ is the weight n Whittaker function.
- $W_{2\pi\omega} : G(\mathbb{R}) \to \mathbb{V}_n^\vee$ is determined by

$$W_{2\pi\omega,n}(x, y, t) = \sum_{-n \leqslant v \leqslant n} \left(\frac{|p_\omega(z)|}{p_\omega(z)} \right)^v t^{2n+2} K_v(2\pi |p_\omega(z)| y^{-3/2}) \frac{x^{n+v} y^{n-v}}{(n+v)!(n-v)!}$$

  along with left N_G-equivariant and right K_G equivariance properties.

Details on the Fourier expansion of QMFs

- When we restrict a QMF F on $G = G_2$ to a QMF $\varphi = F|_H$ on $H = \mathrm{SU}(2,1)$, the Fourier coefficients of φ are finite sums of the Fourier coefficients of F. This is a general phenomenon of QMFs!
- The Fourier expansion of QMFs on H is indexed by elements of $\mathbb{Q}(i)$.
- The *projection* of the binary cubic form $\omega = (a, b/3, c/3, d)$ to E is $\mathrm{pr}(\omega) = (\frac{a-c}{2}, \frac{d-b}{2})$.
- The Fourier coefficient for $\varphi = F|_H$ associated to $\nu = (\frac{a-c}{2}, \frac{d-b}{2})$ is:

$$a_{\varphi,\nu} = \sum_{\mathrm{pr}(\omega)=\nu} a_{F,\omega}.$$

- What about the Fourier coefficients of $\mathcal{D}^n_{r,m} F|_H$? The invariant theory of binary cubic forms comes into play.

Details on the Fourier expansion of QMFs

- For $\omega = (a, b/3, c/3, d)$, let $z_\omega = p_\omega(i) = ai^3 + bi^2 + ci + d$. Note that $\mathrm{pr}(\omega) = \mathrm{pr}(\omega') \implies z_\omega = z_{\omega'}$.
- Let $b_\omega = 2(3ai + b + ci + 3d)$. This is proportional to the square root of the Hessian (quadratic covariant) of the binary cubic form associated to the orthogonal complement of $\mathrm{pr}(\omega)$.
- Define

$$v_\omega = z_\omega(y_s^3) - b_\omega(x_s y_s^2) - \overline{b_\omega}(x_s^2 y_s) + \overline{z_\omega}(x_s^3).$$

Theorem (H.)

Let $\ell^n_{r,m}$ be a (suitably normalized) basis element for $L^n_{r,m}$. Then,

$$(\mathcal{D}^n_{r,m} W^G_{2\pi\omega,n})|_H = \frac{(2\pi)^r}{r!}\langle \ell^n_{r,m}, v^r_\omega\rangle W^H_{2\pi\mathrm{pr}(\omega),n+\frac{r}{2},m}$$

- Proof sketch: Compute the "highest weight" part $D^n_{r,m}\langle W^G_{2\pi\omega,n}, x^{2n}\rangle$ and then appeal to equivariance. Again, a stronger theorem statement is actually needed.

Arithmeticity of Adjoint L-function

Let Π be a cuspidal automorphic representation of $H = \mathrm{SU}(2,1)$ with $\pi_\infty = \pi^H_{\ell,0}$. Let $\varphi \in \Pi$ be a Hecke eigenform. Let $E^G_\ell(g) = E^G_\ell(g, s = \ell + 1)$ be the degenerate Heisenberg Eisenstein series on G_2, with parameter s chosen so that it is a QMF.

- Integral representation: $\langle \varphi, E^G_\ell \rangle_H \approx L_\infty(\ell, \Pi, \mathrm{Ad}) L(\ell, \Pi, \mathrm{Ad})$
- We should be able to write a finite decomposition

$$E^G_\ell|_H = a_0 E^H_\ell + \sum a_i E^H(\chi_i) + \sum b_j f_j$$

  where the $E^H(\chi_i)$ are Eisenstein series on H induced from Hecke characters and the f_j are a basis of eigenforms.
- If we know E^G_ℓ has Fourier coefficients in $\overline{\mathbb{Q}}$, then we can prove all the a_i and b_j are in $\overline{\mathbb{Q}}$.
- Therefore (say $\varphi = f_1$)

$$\langle \varphi, E^G_\ell \rangle_H = a_1 \langle \varphi, \varphi \rangle \in \overline{\mathbb{Q}}$$

- For critical values to the left: replace $E^G_\ell(g, s = \ell + 1)$ with $\mathcal{D}^{\ell-2r}_{2r,0} E^G_{\ell-2r}(g, s = \ell - 2r + 1)$.

More on the arithmeticity of Eisenstein series

Theorem (ongoing joint work with J. Johnson-Leung, F. McGlade, A. Pollack, M. Roy)

The degenerate quaternionic Heisenberg Eisenstein series on G_2 (and $B_3, D_4, F_4, E_6, E_7, E_8$) can be normalized to have algebraic Fourier coefficients.

- To prove the statement for all groups not of type G_2 or D_4, we leverage the *Fourier-Jacobi* expansion, which turns out to be a half-integral weight holomorphic Eisenstein series.
- For G_2 (resp. D_4), we use the pullback procedure described in the previous slide:

  $$E_\ell^{B_3}|_{G_2} = a_0 E_\ell^{G_2} + \sum a_i E^{G_2}(f_i) + \sum b_j F_j.$$

  Now the f_i are actually holomorphic modular forms of weight 3ℓ and the F_j are a basis for cusp forms on G_2.
- There is a basis of cuspidal QMFs, all of whose Fourier coefficients are all algebraic numbers (Pollack).

Thank you!