Arithmeticity of L-functions for Exceptional Groups

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First Example: The Riemann zeta function

Recall the Riemann zeta function $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$.

- Multiplying by a gamma factor $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ yields the completed zeta function
$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$
- Functional equation $\Lambda(s) = \Lambda(1-s)$.
- Call an integer m a critical value for ζ if neither m nor $1-m$ are zeroes or poles of the gamma factor $\pi^{-s/2}\Gamma(s/2)$. The critical values are $m = -1, -3, -5, ..$ and $m = 2, 4, 6, ...$
- For m a positive even integer,
$$\zeta(m) = (-1)^{\frac{m}{2}+1} \frac{(2\pi)^m B_m}{2(m!)}$$
- For example
$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Deligne's Conjecture on Critical Values of L-functions

- General conjecture (Deligne): if L is a motivic L-function, then

$$L(k) \in (\text{period}) \cdot \overline{\mathbb{Q}}$$

  at "critical values".
- One method to prove things about (automorphic) L-functions is to use integral representations and properties of Eisenstein series.
- A lot has been accomplished in this direction, starting with work of Shimura for classical modular forms and then Harris for Siegel modular forms.
- More recently: Horinaga-Pitale-Saha-Schmidt for the standard L-function on $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$, Eischen-Rosso-Shah for the Spin L-function on GSp_6.

Example: Holomorphic Modular Forms

- A modular form f of weight ℓ and level 1 is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying:
  - $f(\gamma(z)) = (cz+d)^{\ell} f(z)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$
  - moderate growth condition
- Modular forms have Fourier expansions:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

  We call f a cusp form if $a_0 = 0$.
- Let f and g be holomorphic modular forms with Fourier expansions

$$f = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, g = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

- Define the product L-function

$$L(s, f \times g) = \sum_{n=0}^{\infty} a_n \overline{b_n} n^{-s}$$

A result of Shimura

Theorem (Shimura)

Let f be a Hecke eigenform of weight ℓ and g a holomorphic modular form of weight $m < \ell$. Then, when k is an integer satisfying $\frac{1}{2}(m + \ell - 2) < k < \ell$,

$$\pi^{-\ell} \frac{L(k, f \times g)}{\langle f, f \rangle} \in \mathbb{Q}(f)\mathbb{Q}(g)$$

- Proof of theorem: Integral representation, control of Fourier coefficients and properties of Eisenstein series, Maass-Shimura operators
- Integral representation (Rankin, Selberg):

$$\langle f(z), g(z) \cdot E_n(z, s) \rangle \approx L(s + \ell - 1, f \times g),$$

  where $E_n(z, s)$ is real-analytic Eisenstein series of weight $n = \ell - m$.
- When $s = 0$, $E_n(z, 0)$ is a holomorphic Eisenstein series of weight n. So

$$\langle f, g \cdot E_n(z, 0) \rangle \approx L(\ell - 1, f \times g),$$

  which implies

$$\pi^{-\ell} \langle f, f \rangle^{-1} L(\ell - 1, f \times g) \in \mathbb{Q}(f)\mathbb{Q}(g).$$

A result of Shimura

- We have the result for the *right-most* critical value in Shimura's theorem.
- To get algebraicity results for critical values to the left of $\ell - 1$, use Maass-Shimura differential operators

$$\delta_n = \frac{1}{2\pi i}\left(\frac{n}{2iy} + \frac{\partial}{\partial z}\right), \delta_n^{(r)} = \delta_{n+2r-2} \circ \cdots \circ \delta_{n+2} \circ \delta_n$$

- Then take a lower weight Eisenstein series of weight $n = \ell - m - 2r$.
- Then $\delta_n^{(r)} E_n(z,0) \approx E_{n+2r}(z,-r)$ and

$$\langle f, g \cdot \delta_n^{(r)} E_n(z,0)\rangle \approx \langle f, g \cdot E_{\ell-m}(z,-r)\rangle \approx L(\ell-1-r, f \times g).$$

- Conclusion: algebraicity of $\pi^{-\ell}\langle f, f\rangle^{-1} L(\ell - 1 - r, f \times g)$

A result of Shimura

- Then $\delta_n^{(r)} E_n(z,0) \approx E_{n+2r}(z,-r)$ and

$$\langle f, g \cdot \delta_n^{(r)} E_n(z,0) \rangle \approx \langle f, g \cdot E_{\ell-m}(z,-r) \rangle \approx L(\ell - 1 - r, f \times g).$$

- Conclusion: algebraicity of $\pi^{-\ell} \langle f, f \rangle^{-1} L(\ell - 1 - r, f \times g)$
- Inner workings: $G = g \cdot \delta_n^{(r)} E_n(z,0)$ is not a modular form, but integrating against it is the same as integrating against a modular form G_0.
- The Fourier coefficients of G_0 lie in $\mathbb{Q}(g)\mathbb{Q}(E_n(z,0))$; this is one thing that the Maass-Shimura operators do for us.
- In fact G_0 is proportional to the *Rankin-Cohen bracket* $[g, E_n]_r$,

$$[g, E_n]_r = (2\pi i)^{-r} \sum_{a+b=r} (-1)^a \binom{m+r-1}{b} \binom{n+r-1}{a} \frac{d^a g}{dz^a} \frac{d^b E_n}{dz^b}.$$

Representation Theory

- We can translate between modular forms and automorphic functions on $G = \mathrm{SL}(2, \mathbb{R})$. If f is a modular form of weight ℓ, define $\phi_f : G \to \mathbb{C}$ via

$$\phi_f(g) = (cz + d)^{-\ell} f(g \cdot i).$$

- Structure of G:
  - Maximal compact subgroup $K = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$
  - Complexified Lie algebra $\mathfrak{g}$ spanned by

$$Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

  - Casimir element in the center of $U(\mathfrak{g})$:

$$\Omega_G = \frac{1}{2}H^2 + XY + YX$$

Representation Theory

- We can translate between modular forms and automorphic functions on $G = \mathrm{SL}(2, \mathbb{R})$. If f is a modular form of weight ℓ, define $\phi_f : G \to \mathbb{C}$ via
$$\phi_f(g) = (cz + d)^{-\ell} f(g \cdot i).$$
- Action of $\mathfrak{g}$ and K:
  - $\phi_f \left(g \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) = e^{-i\ell\theta} \phi_f(g)$
  - For A in $\mathfrak{g}$,
$$A\phi_f(g) := \frac{d}{dt} \phi_f(g \exp(tA))|_{t=0}$$
- f is holomorphic iff $Y\phi_f \equiv 0$.
- $X\phi_f \leftrightarrow (-4\pi)\delta_\ell f$
- ϕ_f generates some (quotient of) the holomorphic discrete series representation π_ℓ

$Y\phi_f = 0$ $\quad \phi_f \quad X\phi_f \quad X^2\phi_f$

Representation Theory

- One can compute (Vergne) the restriction to a diagonally embedded $\mathrm{SL}_2 \subset \mathrm{SL}_2 \times \mathrm{SL}_2$ of the tensor product of two holomorphic discrete series:

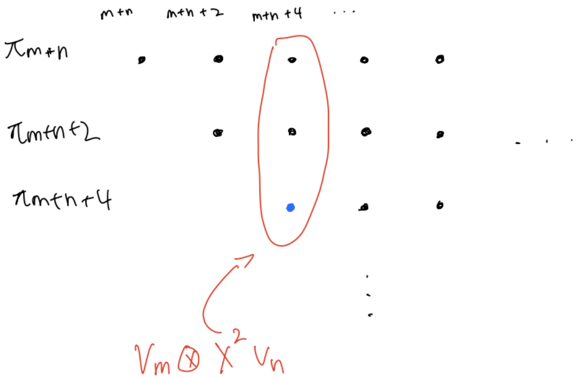
$$\pi_m \otimes \pi_n \cong \bigoplus_{j=0}^{\infty} \pi_{m+n+2j}.$$

- Recall we started with an eigenform f of weight ℓ, a modular form g of weight m, and an Eisenstein series of weight $n = \ell - m - 2r$.
- If g is weight m, then it corresponds to a vector v_m in the lowest K-type of π_m; similarly E corresponds to some $v_n \in \pi_n$.
- We need to compute the projection of $v_m \otimes X^r v_n \in \pi_m \otimes \pi_n$ to π_{m+n+2r} in the above isomorphism.
- Here is one way: compute (up to scalar multiple) the linear combination

$$v = \sum a_j (X^j v_m \otimes X^{r-j} v_n)$$

needed for $\Omega_H v = \lambda_{m+n+2r} v$.

Representation Theory



This process recovers the Rankin-Cohen bracket $[g, E]_r$.

Onto Quaternionic Groups

- There is an integral representation for the order 8 adjoint L-function of $\mathrm{SU}(2,1)$ that uses an Eisenstein series on the exceptional group G_2 (Hundley).
- Both of these groups possess quaternionic discrete series (Gross and Wallach); quaternionic modular forms (QMFs) are automorphic forms whose archimedean component correspond to these.
- QMFs have a robust theory of Fourier expansion and their Fourier coefficients have arithmetic properties (Gan-Gross-Savin, Koseki-Oda, Hilado-McGlade-Yan, Pollack).
- One can hope to apply Shimura's method to prove arithmeticity results for the adjoint L-function of $\mathrm{SU}(2,1)$ in the quaternionic setting.
- In this talk, we focus on the ingredient of differential operators.

Group Theory

- The *split octonions* Θ over $\mathbb{Q}$, for example as defined by the Cayley-Dickson construction:
$$\Theta = \{(x, y) : x, y \in B = M_2(\mathbb{Q})\}.$$
- Let $G = G_2$ be the automorphism group of Θ.
- If $v \in \Theta$ has squarefree norm $D > 0$, then $H = \mathrm{Stab}_G(v)$ is a subgroup of type $\mathrm{SU}(2, 1)$.
- In more detail, $\mathbb{Q}(v) \cong \mathbb{Q}(\sqrt{-D})$ and the orthogonal complement of $\mathbb{Q}(v)$ in Θ is a 3-dimensional $\mathbb{Q}(v)$ Hermitian space.
- For convenience, let $v = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right)$ so that $\mathbb{Q}(v) = \mathbb{Q}(i)$ and $H \hookrightarrow G$ is in "good position".

Group Theory

- The maximal compact subgroups:

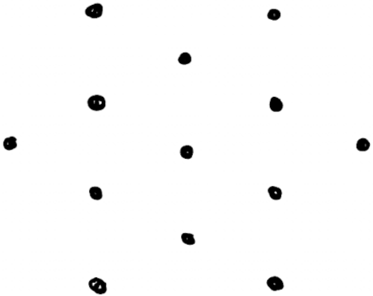
$$K_G := \mathrm{SU}(2)^{\text{long}} \times \mathrm{SU}(2)^{\text{short}}/\mu_2 \supset \mathrm{SU}(2)^{\text{long}} \times U(1)/\mu_2 =: K_H$$

- The Cartan decomposition: Let $\mathfrak{g}$ be the complexified Lie algebra of $G = G_2$. Then

$$\mathfrak{g} = (\mathfrak{sl}_2^{\text{long}} \oplus \mathfrak{sl}_2^{\text{short}}) \oplus \mathfrak{p}$$

  where $\mathfrak{p} \cong V_2^{\text{long}} \oplus \mathrm{Sym}^3(V_2^{\text{short}})$ as a representation of K_G.

Picture and Notation: Root Diagram of G_2



Long root $\mathfrak{sl}_2$: $\{e_u, h_u, f_u\}$

Short root $\mathfrak{sl}_2$: $\{e_s, h_s, f_s\}$

$$\mathfrak{p} = \mathrm{Span}_{\mathbb{C}}\{h_{\pm j}, d_{\pm j}\}$$

$\mathrm{Lie}(H) \otimes \mathbb{C} =$

$$\mathrm{Span}_{\mathbb{C}}\left\{\begin{matrix} e_u, h_u, f_u, h_r, \\ h_{\pm 3}, d_{\pm 3} \end{matrix}\right\}$$

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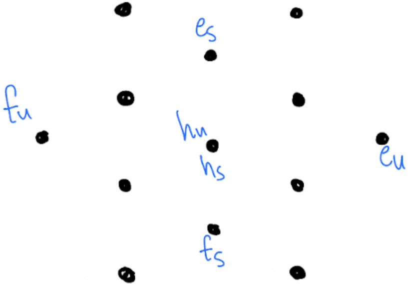

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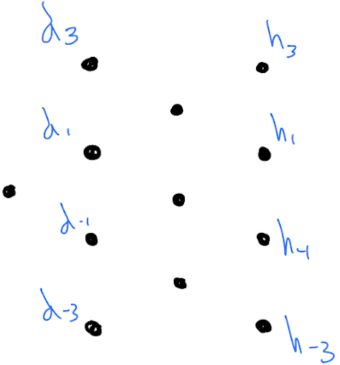

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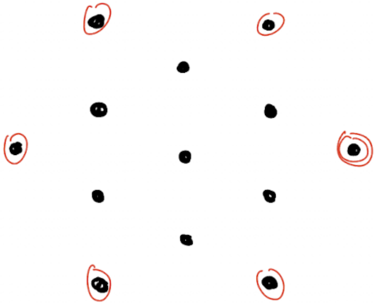

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Quaternionic Modular Forms

- Let $n \geqslant 1$. There is a (limit of) discrete series representation π_n^G of $G = G_2$, with lowest K_G-type $\mathbb{V}_n = \mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathbf{1}$.

Definition (Gan-Gross-Savin, A. Pollack)

A quaternionic modular form (QMF) on $G = G_2$ of weight n is a smooth function $F : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{V}_n^\vee$ satisfying:

1. $F(\gamma g) = \Phi(g)$ for all $\gamma \in G_2(\mathbb{Q})$ and $g \in G_2(\mathbb{A})$
2. $F(gk) = k^{-1}\Phi(g)$ for all $k \in K_G$ and $g \in G_2(\mathbb{A})$
3. $D_n F = 0$ for a certain differential operator D_n

- One can analogously make a definition for QMFs on $H = \mathrm{SU}(2,1)$. These are associated to quaternionic (i.e. "large" or "generic") nonholomorphic discrete series representations $\pi^H_{n+\frac{r}{2},m}$ of H, with lowest K-type $\mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes \mathbb{C}(m)$.

On Discrete Series

The representation π_n^G has K_G-type decomposition

$$\pi_n^G = \bigoplus_{r=0}^{\infty} \mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}})).$$

The representation $\pi^H_{n+\frac{r}{2},m}$ has K_H-type decomposition

$$\pi^H_{n+\frac{r}{2},m} = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n+r+j}(V_2^{\mathrm{long}}) \boxtimes (\mathrm{Sym}^j(\mathbb{C}(-3) \oplus \mathbb{C}(3)) \otimes \mathbb{C}(m)).$$

Theorem (H. Y. Loke)

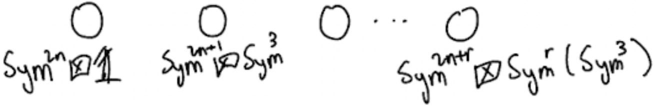
For a nonnegative integer r, let $[r] := \{-r, -r+2, \ldots, r-2, r\}$. Then, as representations of H,

$$\pi_n^G|_H = \bigoplus_{r=0}^{\infty} \bigoplus_{m \in [r]} \pi^H_{n+\frac{r}{2},m}.$$

On Discrete series

The representation π_n^G has K_G-type decomposition

$$\pi_n^G = \bigoplus_{r=0}^{\infty} \mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}})).$$



- By Loke's restriction theorem, there exists a unique line $L^n_{r,m}$ in $\mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}})$ so that

$$\mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes L^n_{r,m} \subseteq \mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{\mathrm{short}}))$$

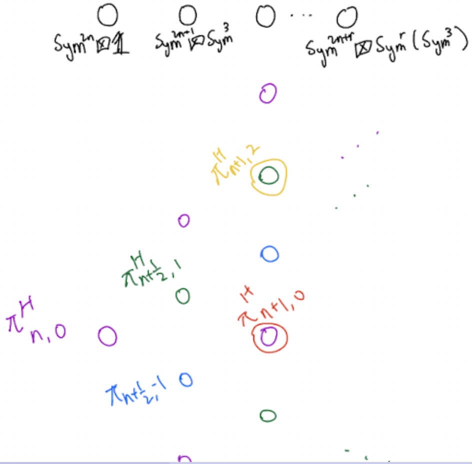
  is the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m}$ in $\pi_n^G|_H$.

On Discrete series

By Loke's restriction theorem, there exists a unique line $L^n_{r,m}$ in $\mathrm{Sym}^r(V_G)$ so that

$$\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^n_{r,m} \subseteq \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)$$

is the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m}$ in $\pi^G_n|_H$.



On Discrete Series

- Goal: Explicitly pin down $L^n_{r,m}$ and how to get there from $\mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathbf{1}$.
- To move between K_G-types: use elements of $\mathfrak{p}^r \subseteq U(\mathfrak{g})$.
- More specifically, we will find find explicit formulas for elements $D^n_{r,m} \in U(\mathfrak{p})$ that take x^{2n} in the lowest K_G-type $\mathrm{Sym}^{2n}(V_2^{\mathrm{long}}) \boxtimes \mathbf{1}$ of π^G_n to to x^{2n+r} in the lowest K_H type of $\pi^H_{n+\frac{r}{2},m}$.
- In applications, this is everything we need, thanks to equivariance properties in the Fourier expansion of a QMF.
- More specifically still, we will construct explicit elements $D^n_{r,m} \in \mathbb{C}[h_{-3}, h_{-1}, h_1, h_3]$ (whose action on x^{2n} is well-defined, e.g. by PBW theorem).

The Highest Weight Operators

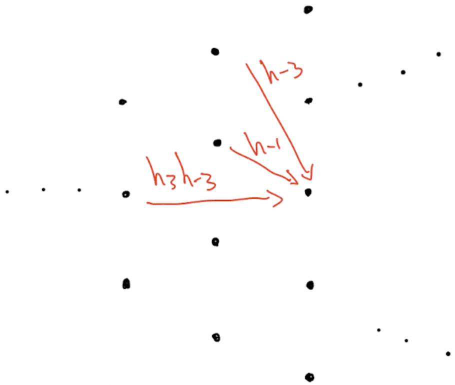
Start with a fixed weight $n \geqslant 1$. Define, for $r \geqslant 0$ and $m \in [r] := \{-r, -r+2, \ldots, r-2, r\}$,

- $A^n_{r,m} := -\dfrac{1}{3}\dfrac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
- $B^n_{r,m} := -\dfrac{1}{9}\dfrac{(4n+r+m-2)(3n+r-2)(n+r-1)(r+m)}{(2n+r+m)(2n+r+m-2)(2n+r-1)(2n+r-2)}$
- $E^n_{r,r} := A^n_{r,-r}$

Definition

Define recursively $D^n_{0,0} = 1, D^n_{1,1} = h_1, D^n_{1,-1} = h_{-1}$, and

- $D^n_{r,m} = h_{-1}D^n_{r-1,m+1} + A^n_{r,m}h_{-3}D_{r-1,m+3} + B^n_{r,m}h_3h_{-3}D^n_{r-2,m}$ for $m < r$.
- $D^n_{r,r} = h_1 D^n_{r-1,m-1} + E^n_{r,r}h_3 D_{r-1,m-3}$



$$D^n_{r,m} = h_{-1} D^n_{r-1,m+1} + A^n_{r,m} h_{-3} D_{r-1,m+3} + B^n_{r,m} h_3 h_{-3} D^n_{r-2,m}$$

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Theorem (H.)

Let x^{2n} be a highest-weight vector in the lowest K_G-type of π^G_n. The vector $D^n_{r,m}x^{2n}$ is a highest-weight vector in the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m} \subseteq \pi^G_n|_H$.

- In order to motivate the proof, and the formulas for $D^n_{r,m}$, we describe an algorithm to explicitly compute $D^n_{r,m}$ for any n, r, m.
- Key: the Casimir element Ω_H acts on $\pi^H_{n+\frac{r}{2},m}$ as a scalar $\lambda^n_{r,m}$ that distinguishes (r, m).

Algorithm

- Start with $v = x^{2n}$ in the lowest K_G-type.
- $D^n_{r,m}$ is a linear combination of $h^a_{-3}h^b_{-1}h^c_1h^d_3$ with $a+b+c+d=r$ and $-3a-b+c+3d=m$. Let
$$X = \sum R_{a,b,c,d}\cdot h^a_{-3}h^b_{-1}h^c_1h^d_3$$
be a general linear combination of such elements.
- Calculate
$$(\Omega_H X - X\Omega_H)v = \left(\sum S_{a,b,c,d}\cdot h^a_{-3}h^b_{-1}h^c_1h^d_3\right)v,$$
where each $S_{a,b,c,d}$ is a $\mathbb{Q}$-linear combination of all the R's.
- Linear algebra problem: Find $R_{a,b,c,d}$ so that
$$(\Omega_H X - X\Omega_H)v = (\lambda^n_{r,m} - \lambda^n_{0,0})Xv.$$
- Then $\Omega_H Xv = \lambda^n_{r,m}Xv$ and we can set $D^n_{r,m}$ to (some normalization of) X. Done!

Remark

This computation boils down to the commutators $[\Omega_H, h^b_{-1}h^c_1]$. Inspecting these is how one can "guess" the shape of the recurrence formulas for $D^n_{r,m}$.

The real theorem statement

- The Algorithm does not suggest a nice closed form formula; furthermore, a recurrence relation is useful for working with the Fourier expansion of a QMF.
- To prove that

$$D^n_{r,m}v = (h_{-1}D^n_{r-1,m+1} + A^n_{r,m}h_{-3}D^n_{r-1,m+3} + B^n_{r,m}h_3h_{-3}D^n_{r-2,m})v$$

  is in the correct piece of $\mathrm{Sym}^{2n+r}(V_2^{\mathrm{long}}) \boxtimes \mathrm{Sym}^r(V_2^{\mathrm{short}})$, one computes the commutator

$$[\Omega_H, h_{-1}] = h_{-1}\left(\frac{2}{3} - \frac{1}{3}h_s + h_u\right) + 2d_{-1}e_u - \frac{2}{3}h_{-3}e_s.$$

- The only problematic term is e_s. Since $e_s \cdot v = 0$, we need to understand $[e_s, D^n_{r,m}]$.
- In the same vein, we need to understand $[f_s, D^n_{r,m}]$.

The real theorem statement

- $A^n_{r,m} := -\frac{1}{3}\frac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
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- $E^n_{r,m} := A^n_{r,-m}$
- $F^n_{r,m} := B^n_{r,-m}$
- $U^n_{r,m} := \frac{1}{2}\frac{(4n+r+m-2)(r+m)}{(2n+r+m-2)}$
- $V^n_{r,m} := \frac{1}{3}\frac{(4n+r-m-2)(3n+r-1)(n+r)(r-m)}{(2n+r-m)(2n+r-m-2)(2n+r-1)}$
- $S^n_{r,m} := U^n_{r,-m}$
- $T^n_{r,m} := V^n_{r,-m}$

The real theorem statement

Theorem (H.)

Let $n \geqslant 1$. Recall the (limit of) discrete series representation π_n^G has lowest K_G-type $\mathbb{V}_n = \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. Let $v = x^{2n} \in \mathbb{V}_n$ be a highest weight vector. For any $r \geqslant 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,

1. *$D^n_{r,m} v$ is an Ω_H-eigenvector with eigenvalue $\lambda^n_{r,m}$, and $D^n_{r,m} v \in x^{2n+r} \boxtimes L^n_{r,m}$ where $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^n_{r,m} \subseteq \pi_n^G$ is the lowest K_H-type of the unique $\pi^H_{n+\frac{r}{2},m} \subseteq \pi_n^G|_H$.*
2. $D^n_{r,m} = h_1 D^n_{r-1,m-1} + E^n_{r,m} h_3 D^n_{r-1,m-3} + F^n_{r,m} h_3 h_{-3} D^n_{r-2,m}$
3. $[f_s, D^n_{r,m}] = U^n_{r,m} D^n_{r,m-2} + V^n_{r,m} h_{-3} D^n_{r-1,m+1}$
4. $[e_s, D^n_{r,m}] = S^n_{r,m} D^n_{r,m+2} + T^n_{r,m} h_3 D^n_{r-1,m-1}$

Algebraicity Results

- Ingredient 1, Integral representation: ✓
- Ingredient 2, Control of Fourier coefficients / properties of Eisenstein series

Theorem (ongoing joint work with J. Johnson-Leung, F. McGlade, A. Pollack, M. Roy)

The degenerate quaternionic Heisenberg Eisenstein series on G_2 (and $B_3, D_4, F_4, E_6, E_7, E_8$) can be normalized to have algebraic Fourier coefficients.

- Cook these up: Let Π be a cuspidal quaternionic representation of $H = \mathrm{SU}(2,1)$. If $\varphi \in \Pi$ is an eigenform, then

$$\frac{L(\ell, \Pi, \mathrm{Ad})}{\langle \varphi, \varphi \rangle} \in \pi^{\mathbb{Z}} \cdot \mathbb{Q}(\varphi).$$

- To get algebraicity results for critical values to the left, we need Ingredient 3, Differential Operators. This is what comes out of all the representation theory discussed.

Exceptional Maass-Shimura Operators

- To get algebraicity results for critical values to the left, we need Ingredient 3, Differential Operators. This is what comes out of all the representation theory discussed.

Corollary (H.)

Let F be a QMF on $G = G_2$ of weight n. For any integers $r \geqslant 0$ and $m \in [r] = \{-r, -r+2, \ldots, r-2, r\}$, there is a differential operator $\mathcal{D}^n_{r,m}$ such that:

- $f = (\mathcal{D}^n_{r,m} F)|_H$ *is a QMF on* $H = \mathrm{SU}(2,1)$ *of weight* $(n + \frac{r}{2}, m)$.
- *The Fourier coefficients of f are $\mathbb{Q}(i)$-linear combinations of the Fourier coefficients of F.*

Thank you!