EXCEPTIONAL MAASS-SHIMURA OPERATORS FOR $\mathrm{SU}(2,1)$ IN G_2

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1. Introduction

The goal of this paper is to define weight-raising differential operators for quaternionic modular forms, and see that they preserve algebraicity of Fourier coefficients. These operators can then be applied to obtain results about special values of L-functions.

Our work has a classical analog in the theory of holomorphic modular forms. In [Shi76], Shimura proves the following theorem:

Theorem 1.0.1 (Shimura)**.** *Let f be a Hecke eigenform of weight ℓ_1 and g a holomorphic modular form of weight $\ell_2 < \ell_1$. Then, when k is an integer with $\frac{1}{2}(\ell_2 + \ell_1 - 2) < k < \ell_1$,*

$$\frac{L(k, f \times g)}{\pi^{\ell_1} < f, f >} \in \mathbb{Q}(f)\mathbb{Q}(g), \textit{ the field generated by the Fourier coefficients of } f \textit{ and } g$$

Shimura's proof utilizes the classical Rankin-Selberg integral representation of $L(s, f \times g)$, in which f is integrated against the product of g and an Eisenstein series. If E_n is a holomorphic Eisenstein series of weight $n = \ell_1 - \ell_2$, then gE_n is a holomorphic modular form of weight ℓ_1. Then $L(\ell_1 - 1, f \times g)$ is closely related to the Petersson inner product $\langle f, gE_n \rangle$, and one leverages known properties of holomorphic Eisenstein series to obtain Shimura's theorem for $k = \ell_1 - 1$. In the context of Deligne's conjecture ([Del79]), this is the right-most critical value of $L(s, f \times g)$.

In order to extend his result to closer-to-central critical values, Shimura utilizes what are now called Maass-Shimura differential operators. These are explicit differential operators $\delta_n^{(r)}$ which take modular forms of weight n to nearly holomorphic modular forms of weight $n + r$. In order to access $L(\ell_1 - 1 - r, f \times g)$, one starts with a weight $n = \ell_1 - \ell_2 - r$ holomorphic Eisenstein series E_n and then integrates f against $g \cdot \delta_n^{(r)} E_n$. While $g \cdot \delta_n^{(r)} E_n$ is no longer a classical modular form, Shimura proves that there is some holomorphic modular form g_0 such that $\langle f, g \cdot \delta_n^{(r)} E_n \rangle = \langle f, g_0 \rangle$. Furthermore, the properties of $\delta_n^{(r)}$ and the structure theory of nearly holomorphic modular forms imply that the Fourier coefficients of g_0 lie in $\mathbb{Q}(g)\mathbb{Q}(E_n)$.

Shimura's technique has since been expanded and carried out in many higher rank situations, starting with Harris's work on scalar-valued Siegel modular forms in [Har81], to recent results such as the standard L-function of vector-valued Siegel modular forms ([Hor+22]), and the spin L-function for GSp_6 ([ERS24]); our list of the interesting work done in this area is far from complete.

In contrast to these previous results, we work with a class of automorphic forms that are not holomorphic. The examples referenced above all involve holomorphic modular forms (or Siegel modular forms or Hermitian modular forms). These correspond to automorphic forms on groups such as SL_2 or Sp_{2n} which admit holomorphic discrete series representations. There is a class of groups, the quaternionic groups, which do not necessarily have holomorphic discrete series but do have quaternionic discrete series as studied by Gross and Wallach in [GW96].

On these groups, one can define quaternionic modular forms (QMFs). Roughly speaking, QMFs are automorphic forms whose archimedean component lie in some quaternionic discrete series. The Fourier coefficients of QMFs and their arithmetic properties were first studied for the group G_2 by Gan-Gross-Savin ([GGS02]); Pollack developed a robust theory of Fourier expansion for QMFs for a larger family of groups including all exceptional quaternionic groups ([Pol21]) and established the algebraicity of cusp forms in ([Pol24]).

Two examples of quaternionic groups are $\mathrm{SU}(2,1)$ and the exceptional group G_2. In fact one can embed $\mathrm{SU}(2,1) \hookrightarrow G_2$. Hundley ([Hun12]) uses this embedding in an integral representation for the adjoint L-function of a cuspidal automorphic representation Π of $\mathrm{SU}(2,1)$. More precisely, Hundley proves that integrating $f \in \Pi$ against a certain degenerate Eisenstein series E on G_2 represents $L(s, \Pi, \mathrm{Ad})$. When we take f and E to be QMFs, we can precisely calculate the archimedean integral and say something about the special values of $L(s, \Pi, \mathrm{Ad})$. We remark that even though $\mathrm{SU}(2,1)$ has holomorphic modular forms, this exact technique does not apply in that context. Hundley's integral requires generic automorphic representations, and therefore vanishes for holomorhpic modular forms on $\mathrm{SU}(2,1)$. It is a general phenomenon that, like in Shimura's classical result, this technique will only grant us access to the right-most critical value. We develop a theory of exceptional Maass-Shimura operators in order to access closer-to-central critical values. Our main result is the following:

Theorem 1.0.2. *Let $n \geq 1$. For any integers $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$, there exist differential operators $\mathcal{D}^n_{r,m}$ with the following properties:*

- *If Φ is a QMF on G_2 of weight n, then $\varphi = \mathcal{D}^n_{r,m}\Phi|_{SU(2,1)}$ is a QMF on $SU(2,1)$ of weight $(n + \frac{r}{2}, m)$;*
- *the Fourier coefficients of φ are $\overline{\mathbb{Q}}$-linear combinations of the Fourier coefficients of Φ.*

Compared to the ingredients of Theorem 1.0.1, our operators $\mathcal{D}^n_{r,m}$ are analogous to the combined steps of the Maass-Shimura operator and then isolating the g_0 component. In [Har79], Harris interprets these ingredients in representation theoretic terms. Taking this as inspiration, the starting point for our work is a branching problem for quaternionic discrete series on G_2 restricted to $\mathrm{SU}(2,1)$. Restriction problems of this type are solved by Loke ([Lok99]). So for us, the real work lies not in proving the existence of $\mathcal{D}^n_{r,m}$, but rather in finding their explicit formulae and seeing that they preserve algebraicity of Fourier coefficients.

1.1. **Organization.** Sections 2 and 3 are preliminaries. In Section 4 we define our differential operators and find an explicit recurrence formula for their "highest weight" components. We establish algebraic and analytic properties of these highest weight operators. Theorem 1.0.2 can be deduced from these properties, as explained in Section 5.

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2. Groups and Embeddings, Characters, Notation

In this section we set up the notation that we will use throughout the paper.

2.1. **The Group G_2 and its Lie algebra.** For more details about the split group of type G_2 and its Lie algebra, we refer to [Pol21] and [PWZ19], whose exposition and notation (with minor changes) we follow.

2.1.1. *The octonions and G_2.* The discussion here is valid over any field F of characteristic 0, but we will fix $F = \mathbb{Q}$. We begin by recalling the Zorn model of the split octonions over $\mathbb{Q}$. Let $V_3/\mathbb{Q}$ be the standard representation of SL_3 and $V_3^\vee$ the dual representation. Then we have an isomorphism $\wedge^3 V_3 \to \mathbb{Q}$. Fix a basis $\{e_1, e_2, e_3\}$ of V_3 with corresponding dual basis $\{e_1^*, e_2^*, e_3^*\}$ so that $e_1 \wedge e_2 \wedge e_3 \mapsto 1 \in \mathbb{Q}$ in $\wedge^3 V_3 \cong \mathbb{Q}$. With this identification we also have $\wedge^2 V_3 \cong V_3^\vee$ and $\wedge^2 V_3^\vee \cong V_3$, for example $e_1 \wedge e_2 = e_3^*$ and $e_1^* \wedge e_2^* = e_3$.

We recall the Zorn model Θ of the split octonions. Let

$$\Theta = \left\{ \begin{pmatrix} a & v \\ \phi & d \end{pmatrix} : a, d \in \mathbb{Q}, v \in V, \phi \in V^\vee \right\}.$$

Given $x = \begin{pmatrix} a & v \\ \phi & d \end{pmatrix} \in \Theta$, we can define:

- its conjugate $x^* = \begin{pmatrix} d & -v \\ -\phi & a \end{pmatrix}$;
- its norm $N(x) = ad - \phi(v)$;
- its trace $\mathrm{Tr}(x) = a + d$.

Suppose $x = \begin{pmatrix} a & v \\ \phi & d \end{pmatrix}$ and $y = \begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix}$ are two elements of Θ. There is a non-degenerate symmetric bilinear form $(\ , \ )$ on Θ defined by $(x, y) = N(x + y) - N(x) - N(y)$. We can compute

$$(x, y) = aa' + dd' - \phi(v') - \phi'(v).$$

The multiplication of elements in Θ is described by

$$\begin{pmatrix} a & v \\ \phi & d \end{pmatrix} \cdot \begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix} = \begin{pmatrix} aa' + \phi'(v) & av' + d'v - \phi \wedge \phi' \\ a'\phi + d\phi' + v \wedge v' & dd' + \phi(v') \end{pmatrix}.$$

We will sometimes abuse notation and write a for $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \Theta$ and identify elements of V_3 or V_3^* with elements of Θ, for example $e_1 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \in \Theta$.

Define the linear algebraic group G over $\mathbb{Q}$ by

$$G(\mathbb{Q}) = \{ g \in \mathrm{GL}(\Theta) : g(xy) = g(x)g(y) \text{ for all } x, y \in \Theta \}.$$

Then G is a split group of type G_2. From the definition, any $g \in G(\mathbb{Q})$ has the additional properties $g1 = 1$ and $N(gx) = N(x)$ and $\mathrm{Tr}(gx) = \mathrm{Tr}(x)$ for any $x \in \Theta$.

2.1.2. *The Lie algebra of G_2.* Let $V_7 \subseteq \Theta$ be the 7-dimensional subspace of trace 0 elements. In fact V_7 is orthogonal to 1 with respect to the bilinear form on Θ. Then for any $x \in \Theta$, define its imaginary part $Im(x) = \frac{1}{2}(x - x^*)$. The decomposition of x into a "real" and "imaginary" part is the same as its orthogonal decomposition, i.e.

$$x = \frac{1}{2}\operatorname{Tr}(x) \cdot 1 + Im(x)$$

and $Im(x) \in V_7$. Since G fixes 1, we have a representation $G \to \mathrm{GL}(V_7)$ which in fact factors through $\mathrm{SO}(V_7)$. The Lie algebra of $\mathrm{SO}(V_7)$ can be identified with $\wedge^2 V_7$. In this identification, the action of $w \wedge x$ on $v \in V_7$ is

$$(w \wedge x) \cdot v = (x, v)w - (w, v)x,$$

the Lie bracket is

$$[w \wedge x, y \wedge z] = (x, y)w \wedge z - (x, z)w \wedge y - (w, y)x \wedge z + (w, z)x \wedge y,$$

and the Killing form is proportional to the pairing

$$(w \wedge x, y \wedge z) = (w, z)(x, y) - (w, y)(x, z).$$

There is an alternating map $\wedge^2 V_7 \mapsto V_7$ given by $w \wedge x \mapsto Im(wx)$. The kernel of this map, $\mathfrak{g}_0$, turns out to be the Lie algebra of G. Viewing $\mathfrak{g}_0 \subseteq \wedge^2 V_7$ is useful for computations.

Abstractly, there is the $\mathbb{Z}/3$-grading $\mathfrak{g}_0 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee$. We will give an explicit basis for $\mathfrak{g}_0$, as elements of $\wedge^2 V_7$, that corresponds to this $\mathbb{Z}/3$-grading. Let $u_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in V_7$. Define, with indices in $\{1, 2, 3\}$ taken modulo 3:

- $E_{kj} = e_j^* \wedge e_k$
- $v_j = u_0 \wedge e_j + e_{j+1}^* \wedge e_{j+2}^*$
- $\delta_j = u_0 \wedge e_j^* + e_{j+1} \wedge e_{j+2}$.

One can check that v_j and δ_j are in $\mathfrak{g}_0$ for all j, and $E_{jk} \in \mathfrak{g}_0$ if $j \neq k$. Furthermore,

$$\mathfrak{h} = \{\alpha_1 E_{11} + \alpha_2 E_{22} + \alpha_3 E_{33} : \alpha_1 + \alpha_2 + \alpha_3 = 0\}$$

is a Cartan subalgebra for $\mathfrak{g}_0$. The v_j, δ_j, E_{jk} with $j \neq k$ along with $\mathfrak{h}$ span all of $\mathfrak{g}_0$. As suggested by the notation, the E_{jk} correspond to the standard basis for $\mathfrak{sl}_3$ in the $\mathbb{Z}/3$-grading, while $\{v_1, v_2, v_3\}$ is a basis for the V_3 with dual basis $\{\delta_1, \delta_2, \delta_3\}$. The Lie bracket of an element of $\mathfrak{sl}_3$ with a v_j or δ_j is given by the standard or dual representation; The Lie bracket between other elements is computed as:

- $[\delta_{j-1}, v_j] = 3E_{j,j-1}$
- $[v_{j-1}, \delta_j] = -3E_{j-1,j}$
- $[\delta_{j-1}, \delta_j] = 2v_{j+1}$
- $[v_{j-1}, v_j] = 2\delta_{j+1}$
- $[\delta_j, v_j] = 3E_{jj} - (E_{11} + E_{22} + E_{33})$.

There is a natural choice of Cartan involution on G induced from an involution on Θ. Namely, let $\iota : \Theta \to \Theta$ be given by

$$\iota\left(\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}\right) = \begin{pmatrix} d & -\tilde{\phi} \\ -\tilde{v} & a \end{pmatrix}$$

where $\tilde{e_j} = e_j^*$ and $\tilde{e_j^*} = e_j$, and $\tilde{\cdot}$ is extended linearly to the rest of V_3 and $V_3^\vee$. Corresponding to ι is an involution θ on $\wedge^2 V_7$, given by $\theta(w \wedge x) = \iota(w) \wedge \iota(x)$. One can check that θ preserves $\mathfrak{g}_0$ and is a Cartan involution on $\mathfrak{g}_0$.

Let $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k}$ the $+1$ eigenspace of θ in $\mathfrak{g}$ and $\mathfrak{p}$ the -1 eigenspace. In fact, $\mathfrak{k} \cong \mathfrak{sl}_2^{long} \oplus \mathfrak{sl}_2^{short}$, with the superscripts indicating whether the $\mathfrak{sl}_2$ corresponds to a long or short root. On the group level, the corresponding maximal compact subgroup K_G of $G(\mathbb{R})$ is isomorphic to $SU(2)^{long} \times SU(2)^{short} / < (-I, -I) >$. Let V_2^{long} be the standard representation of $SU(2)^{long}$; we take $\{x, y\}$ as a standard basis. Similarly define V_2^{short} and $\{x_s, y_s\}$. Let $V_G = \mathrm{Sym}^3(V_2^{short})$, a 4-dimensional irreducible representation of $SU(2)^{short}$. We fix symplectic forms on V_2^{long} via $\langle x, y\rangle = 1$ and on V_G via $\langle x_s^3, y_s^3\rangle = 1, \langle x_s^2 y_s, x_s y_s^2\rangle = -1/3$ to identify these representations (and their symmetric powers) with their duals.

One can realize $\mathfrak{p} \cong V_2^{long} \boxtimes V_G$ as a representation of K_G. In [Pol21] section 4.1, explicit formulas are given for basis elements of $\mathfrak{k}$ and $\mathfrak{p}$. We use this basis, except we write the subscript s instead of r for the short root $\mathfrak{sl}_2$ in $\mathfrak{k}$. So $\mathfrak{k} = \mathrm{Span}_{\mathbb{C}}\{e_u, f_u, h_u, e_s, h_s, f_s\}$, where the u-subscripts denote a long root $\mathfrak{sl}_2$ triple and the s-subscripts denote a short root $\mathfrak{sl}_2$ triple. And $\mathfrak{p} = \mathrm{Span}_{\mathbb{C}}\{h_j, d_j : j = \pm 1, \pm 3\}$, where in realizing $\mathfrak{p}$ as a representation of K_G, the $h_{3-2j} = x \boxtimes x_s^{3-j} y_s^j$ and the $d_{3-2j} = y \boxtimes x_s^{3-j} y_s^j$.

2.2. **Embedding** SU(2, 1) **inside** G_2.

2.2.1. *A Hermitian subspace of* Θ.

Let $D > 0$ be a square-free integer. Let $v_D = e_2 - De_2^* \in \Theta$. Then $K = \mathrm{Span}_{\mathbb{Q}}\{1, v_D\} \subseteq \Theta$ is a subalgebra isomorphic to $\mathbb{Q}(\sqrt{-D})$, i.e. a embedding from $\mathbb{Q}(\sqrt{-D}) \hookrightarrow \Theta$ is given by $1 \mapsto 1$ and $\sqrt{-D} \mapsto v_D$. We remark that the conjugation on Θ restricts to complex conjugation on $\mathbb{Q}(\sqrt{-D})$.

Let V_D be the orthogonal complement of K in Θ, i.e. $\Theta = V_D \oplus \mathrm{Span}_{\mathbb{Q}}\{1, v_D\}$ is an orthogonal decomposition relative to the bilinear form on Θ. A basis for V_D is

$$V_D = \mathrm{Span}_{\mathbb{Q}}\{u_0, e_2 + De_2^*, e_1, e_3^*, e_1^*, e_3\}.$$

One can then check directly that $(e_2 - De_2^*)V_D = V_D$. So we may endow V_D with the structure of a 3-dimensional (left) K-vector space, with Hermitian form

$$\mathcal{H}(x, y) = \mathrm{Tr}(xy) - \frac{1}{\sqrt{-D}}\,\mathrm{Tr}(x v_D y).$$

Let us check that this is actually a Hermitian form. The difficulty lies in linearity with respect to $v_D \approx \sqrt{-D}$. First

$$\begin{aligned}\mathcal{H}(x, v_D y) &= \mathrm{Tr}(xv_D y) - \frac{1}{\sqrt{-D}}\,\mathrm{Tr}(xv_D v_D y)\\ &= \mathrm{Tr}(xv_D y) - \sqrt{-D}\,\mathrm{Tr}(xy)\\ &= -\sqrt{-D}\cdot\mathcal{H}(x,y)\end{aligned}$$

so it is conjugate linear in the second term. Next in

$$\mathcal{H}(v_D x, y) = \mathrm{Tr}(v_D xy) - \frac{1}{\sqrt{-D}}\,\mathrm{Tr}(v_D x v_D y)$$

we must check for example that $\mathrm{Tr}(v_D xy) = -\mathrm{Tr}(xv_D y)$. For this, we need the general identity for $x, z \in \Theta$ that

$$xz + zx = \mathrm{Tr}(x)z + \mathrm{Tr}(z)x - (x, z).$$

Therefore

$$\mathrm{Tr}(v_D xy) = \mathrm{Tr}(xv_D y) = \mathrm{Tr}(x)\,\mathrm{Tr}(v_D y) + \mathrm{Tr}(v_D)\,\mathrm{Tr}(xy) - (x, v_D)\,\mathrm{Tr}(y) - \mathrm{Tr}(xv_D y)$$

but $\mathrm{Tr}(v_D) = 0$, and $\mathrm{Tr}(v_D y) = (v_D, y^*) = 0 = (x, v_D)$ when $x, y \in V_D$. It is a similar check that $\mathrm{Tr}(v_D x v_D y) = D\,\mathrm{Tr}(xy)$.

2.2.2. *The group $SU(2,1)$ and its Lie algebra.*

Let $H \subseteq G$ be the stabilizer of v_D. Then for any $h \in H$ and $x, y \in V_D$, we have $(hx, v_D) = (hx, hv_D) = (x, v_D) = 0$ and similarly $\mathcal{H}(hx, hy) = \mathcal{H}(x, y)$. We see that H is a group of type $SU(2,1)$. We can calculate that the Lie algebra of H, i.e. the subspace of $\mathfrak{g}_0$ that annihilates v_D, has a basis

$$\{E_{11} - E_{33}, v_2 + D\delta_2, \delta_3 - DE_{12}, E_{23} - Dv_1, E_{13}, E_{21} + Dv_3, \delta_1 + DE_{32}, E_{31}\}.$$

From now on fix $D = 1$. This choice ensures that H is embedded in G in “good position", i.e. the choice we made for the Cartan involution on G restricts to one for H. In fact the “good basis” for $\mathfrak{k}$ and $\mathfrak{p}$ we took from [Pol21] contains a good basis for the Cartan decomposition of H. We have $\mathrm{Lie}(H) \otimes \mathbb{C} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ with

$$\begin{aligned}\mathfrak{k}_H &= \mathrm{Span}_{\mathbb{C}}\{e_u, h_u, f_u, h_s\}\\ \mathfrak{p}_H &= \mathrm{Span}_{\mathbb{C}}\{h_{-3}, h_3, d_{-3}, d_3\}.\end{aligned}$$

The maximal compact of H is $K_H \cong U(2) \cong SU(2)^{long} \times U(1)/ < (-I, -I) >$. Let $V_H \subseteq V_G$ be the subspace $\mathrm{Span}_{\mathbb{C}}\{(y_s^3), (x_s^3)\}$. Then $\mathfrak{p}_H \cong V_2^{long} \boxtimes V_H \cong V_2 \boxtimes (\det^{-3} \oplus \det^3)$ as a representation of K_H.

2.3. **Characters and the Heisenberg parabolic.** We fix as a standard additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ with $\psi(a_\infty) = e^{-2\pi i a_\infty}$ for $a_\infty \in \mathbb{R}$.

2.3.1. *The Heisenberg Parabolic.* Let P_G be the stabilizer in G of E_{13} under the adjoint representation. This is the Heisenberg parabolic subgroup; in its Levi decomposition $P_G = M_G N_G$, one has $M_G \cong \mathrm{GL}_2$ and $Z(N_G) = [N_G, N_G]$. In particular, $\mathrm{Lie}(M_G)$ is spanned by $\{\delta_2, v_2\}$ and $\mathfrak{h}$; an explicit identification of M_G with GL_2 is given in [Pol21] section 2.3. Also $\mathrm{Lie}(N_G)$ is spanned by $\{E_{12}, v_1, \delta_3, E_{23}, E_{13}\}$, with $Z(N_G)$ the one-dimensional space generated by E_{13}.

Then $P_H = H \cap P_G$ is a Heisenberg parabolic (aka Borel, in this case) subgroup of H. We have the Levi decomposition $P_H = M_H N_H$, with $M_H = H \cap M_G$ and $N_H = H \cap N_G$. The Lie algebra of M_H is spanned by $\{E_{11} - E_{33}, v_2 + D\delta_2\}$ and the Lie algebra of N_H is spanned by $\{\delta_3 - DE_{12}, E_{23} - Dv_1, E_{13}\}$. Note that $M_H(\mathbb{R}) = S_H(\mathbb{R}) \times A_H(\mathbb{R}) \cong U(1) \times \mathrm{GL}_1(\mathbb{R})$ in its decomposition into a compact and split part. The $\mathrm{GL}_1(\mathbb{R})$ is generated by $E_{11} - E_{33}$, and the $U(1)$ is generated by $v_2 + \delta_2$.

2.3.2. *Characters.* Define $W_G := N_G^{ab} = N_G/[N_G, N_G]$. Characters χ on $N_G(\mathbb{Q})\backslash N_G(\mathbb{A})$ correspond to elements $\omega \in W_G(\mathbb{Q})$. We will make this explicit.

For any $\mathbb{Q}$-algebra R, we fix a bijection of $W_G(R)$ with the space of binary cubic forms over R as

$$aE_{12} + \frac{b}{3}v_1 + \frac{c}{3}\delta_3 + dE_{23} \in W_G \longleftrightarrow au^3 + bu^2v + cuv^2 + dv^3.$$

Write $\left(a, \frac{b}{3}, \frac{c}{3}, d\right)$ to denote $au^3 + bu^2v + cuv^2 + dv^3$ or its corresponding element of W_G. The adjoint action of M_G on $N_G/[N_G, N_G]$ corresponds to the representation of M_G on the space of binary cubic forms given by

$$(m \cdot f)\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \det(m)^2 f\left(m^{-1}\begin{pmatrix} u \\ v \end{pmatrix}\right).$$

There is a symplectic form on W_G given by

$$\left\langle \left(a, \frac{b}{3}, \frac{c}{3}, d\right), \left(a', \frac{b'}{3}, \frac{c'}{3}, d'\right) \right\rangle = ad' - \frac{1}{3}bc' + \frac{1}{3}cb' - da'.$$

This form is preserved, up to similitude, by M_G.

The image of any $X \in N_G$ modulo $[N_G, N_G]$ is some $p_X = \left(a', \frac{b'}{3}, \frac{c'}{3}, d'\right) \in W_G$. Then associated to $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in W_G(\mathbb{Q})$ is the character $\chi_\omega : N_G(\mathbb{Q})\backslash N_G(\mathbb{A}) \to \mathbb{C}^\times$ given by $\chi_\omega(X) = \psi(\langle \omega, p_X \rangle)$. Also, for $\omega \in W_G(\mathbb{R})$, we will slightly abuse notation and write $p_\omega(z)$ for the polynomial $az^3 + bz^2 + cz + d$ and $p_\omega(u, v)$ for the binary cubic form $au^3 + bu^2v + cuv^2 + d$.

Under these identifications, elements μ of $W_H := N_H^{ab}$ correspond to tuples of the form $\mu = (a, -d, -a, d)$, which correspond to characters χ_μ of $N_H(\mathbb{Q})\backslash N_H(\mathbb{A})$ via $\chi_\mu(X) = \psi(\langle \mu, p_X \rangle)$. Write (a, d) for the element $(a, -d, -a, d) \in W_H$ and $p_\mu(z)$ for the polynomial $az^3 - 3dz^2 - 3az + d$. We remark that any such element is "positive" in the sense of [Pol20] as long as it is nonzero. Indeed, one can compute that the discriminant of $p_\mu(z)$ is $108(a^2 + d^2)^2$.

Now let $pr : W_G \to W_H$ be given by

$$\left(a, \frac{b}{3}, \frac{c}{3}, d\right) \mapsto \left(\frac{a-c}{4}, \frac{d-b}{4}\right).$$

This projection is the identity on W_H; the projection onto the orthogonal complement is given by

$$pr^\perp\left(\left(a, \frac{b}{3}, \frac{c}{3}, d\right)\right) = \left(\frac{3a+c}{4}, \frac{3d+b}{12}, \frac{3a+c}{12}, \frac{3d+b}{4}\right).$$

Let $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in W_G(\mathbb{R})$. When we restrict a Fourier coefficient on G_2 to $\mathrm{SU}(2,1)$, the quantities $p_\omega(i)$ will be involved. However we have $p_{pr(\omega)}(i) = p_\omega(i)$ and $p_{pr^\perp(\omega)}(i) = 0$. One way to distinguish different ω at the level of $\mathrm{SU}(2,1)$ is related to the quadratic covariant of binary cubic forms. Namely, define the binary quadratic form

$$q_\omega(u,v) = (b^2-3ac)u^2 + (bc-9ad)uv + (c^2-3bd)v^2,$$

and let $q_\omega(z)$ be the associated quadratic polynomial. Then q vanishes on W_H, i.e. $q_{pr(\omega)}(i)=0$ for any ω. On the other hand,

$$q_{pr^\perp(\omega)}(i) = -\frac{1}{4}(3ai+b+ci+3d)^2$$

is nonzero as long as $\omega \notin W_H$. These quantities appear in the Fourier coefficients of our differentiated QMFs.

Definition 2.3.1. For $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in W_G(\mathbb{R})$, define the quantities

- $z_\omega := 2p_\omega(i) = 2p_{pr(\omega)}(i) = -2ai - 2b + 2ci + 2d$
- $b_\omega := -4ip'_\omega(i) + 6p_\omega(i) = 6ai + 2b + 2ci + 6d.$

We remark that $b_\omega(i)^2 = -16q_{pr^\perp(\omega)}(i)$ and $b_\omega(i)=0$ if $\omega \in W_H$.

3. Quaternionic Modular Forms and their Fourier Expansion

3.1. **Quaternionic Discrete Series.**

3.1.1. *G_2.* For any integer $n \geq 2$, there is a discrete series representation π_n^G of $G(\mathbb{R})$ studied by Gross and Wallach in [GW96], with K_G-type decomposition

$$\pi_n^G|_{K_G} = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G).$$

Write $\mathbb{V}_n^G = \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$ for the lowest K_G-type. We remark that there is also a limit discrete series at $n=1$ with the same K_G-type decomposition.

It is quite complicated in general to figure out how $U(\mathfrak{p})$ moves around vectors between K_G-types. All we will need to keep in mind is the following: for $v \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$ the lowest K_G-type of π_G^n, the map

$$\begin{aligned}\mathfrak{p}^{\otimes r} \otimes (\mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}) \to (\mathrm{Sym}^r(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)) \otimes (\mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}) \\ \to \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)\end{aligned}$$

must just be multiplication on elements. For example, for $x^{2n} \in \mathrm{Sym}^{2n}(V_2^{long})$ and $h_3 d_{-3} \in \mathfrak{p}^{\otimes 2}$, it takes some work to figure out completely the decomposition of

$$h_3 d_{-3} x^{2n} \in \bigoplus_{j=0}^{2} \mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G),$$

but on the other hand we can read off that the projection of $h_3 d_{-3} x^{2n}$ to the top piece $\mathrm{Sym}^{2n+2}(V_2^{long}) \boxtimes \mathrm{Sym}^2(V_G)$ is equal to $x^{2n+1}y \boxtimes (y_s^3)(x_s^3)$.

3.1.2. *$SU(2,1)$ and Restriction.* For any integer $n \geq 1$, $\epsilon \in \{0,1\}$, and integer m with the same parity as ϵ, there is a discrete series representation $\pi^H_{n+\frac{\epsilon}{2},m}$ with K_H-type decomposition

$$\pi^H_{n+\frac{\epsilon}{2},m}|_{K_H} = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n+\epsilon+j}(V_2^{long}) \boxtimes (\mathrm{Sym}^j(V_H) \otimes \det^m).$$

Write $\mathbb{V}^H_{n+\frac{\epsilon}{2},m} = \mathrm{Sym}^{2n+\epsilon}(V_2^{long}). \boxtimes \det^m$ for the lowest K_H-type. These are the "large" discrete series of $\mathrm{SU}(2,1)$, and are in general neither holomorphic nor antiholomorphic.

Corollary 4.2.2 of [Lok99], describes the restriction of π^G_n to H. We get

$$\pi^G_n|_H = \bigoplus_{r=0}^{\infty} \bigoplus_{\substack{m=-r \\ m\equiv r(2)}}^{r} \pi^H_{n+\frac{r}{2},m}.$$

In particular, the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m}$ in $\pi^G_n|_H$ is $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L_n^{r,m}$ for a particular line in $L_n^{r,m} \subseteq \mathrm{Sym}^r(V_G)$ that has h_s-eigenvalue equal to m. Therefore in the K_G-type decomposition of $\pi^G_n|_H$, each $\mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G)$ comes from $\pi^H_{n+\frac{r}{2},m}$ for $r \le j$. More precisely,

$$\mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G) = \bigoplus_{j=0}^{r} \bigoplus_{\substack{-j\le m\le j \\ m\equiv r(2)}} \mathfrak{p}_H^{\otimes(r-j)} \cdot \mathbb{V}^H_{n+\frac{j}{2},m}.$$

The point of this discussion is to help us fix a particular basis element $\ell^n_{r,m}$ for each $L^n_{r,m}$. The line $L^n_{r,m}$ is spanned by an element of the form

$$v^n_{r,m} = \sum_{\substack{0\le a,b,c,d\le r \\ -3a-b+c+3d=m}} C_{a,b,c,d}(y_s^3)^a(x_sy_s^2)^b(x_s^2y_s)^c(x_s^3)^d.$$

When $(a,b,c,d) = (0,(r-m)/2,(r+m)/2,0)$, we must have $C_{a,b,c,d} \neq 0$. Otherwise, $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathbb{C}v^n_{r,m}$ would come from lower weight H-discrete series; it would be in the image of

$$\bigoplus_{j=0}^{r-1} \bigoplus_{\substack{-j\le m\le j \\ m\equiv j(2)}} \mathfrak{p}_H^{\otimes(r-j)} \cdot \mathbb{V}^H_{n+\frac{j}{2},m}.$$

We define $\ell^n_{r,m} = (C_{a,b,c,d})^{-1}v^n_{r,m}$, i.e. by normalizing its $(0,(r-m)/2,(r+m)/2,0)$ coefficient.

For $U(\mathrm{Lie}(H) \otimes \mathbb{C}) \subseteq U(\mathfrak{g})$, we take as a quadratic Casimir element

$$\Omega_H := \frac{1}{6}h_s^2 + \frac{1}{2}h_u^2 + (e_uf_u + f_ue_u) + \frac{1}{16}(h_3d_{-3} + d_{-3}h_3) - \frac{1}{16}(h_{-3}d_3 + d_3h_{-3}).$$

Then, one can check (see e.g. *loc. cit.* Theorem 3.3.1 for the infinitesimal character) that Ω_H acts on $\pi^H_{n+\frac{r}{2},m}$ as the scalar $\lambda^n_{r,m} := \frac{m^2}{6} + 2(n+\frac{r}{2})^2 - 2$.

3.2. **Quaternionic Modular Forms.**

3.2.1. G_2. Quaternionic modular forms for G_2 were first studied by Gan-Gross-Savin in [GGS02]. We give a slightly broader definition following Pollack [Pol20].

We first define an operator $\mathcal{D}_n$ on functions $\Phi : G(\mathbb{A}) \to \mathbb{V}_n^\vee$ as follows. Let $\tilde{D}_G$ be the operator defined by

$$\tilde{D}_G\Phi = \sum_{X_i} X_iF \otimes X_i^*$$

where the sum is over a basis $\{X_i\}$ of $\mathfrak{p}$ and $\{X_i^*\}$ is the corresponding dual basis. There is a K_G-equivariant projection $pr_- : \mathbb{V}_n^\vee \otimes \mathfrak{p}^\vee \to \mathrm{Sym}^{2n-1}(V_2^{long})^\vee \boxtimes V_G^\vee$. Finally let $\mathcal{D}_n = pr_- \circ \tilde{D}_G$.

Now we can define QMFs for G:

Definition 3.2.1. Let $n \geq 1$ be an integer and $\Phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{V}_n^\vee$. We say that Φ is a modular form on G of weight n if

(1) $\Phi(gk) = k^{-1}\Phi(g)$ for all $g \in G(\mathbb{A})$ and $k \in K_G$
(2) $\mathcal{D}_n\Phi \equiv 0$
(3) Φ is smooth, moderate growth, and $Z(\mathfrak{g})$-finite.

3.2.2. $SU(2,1)$. Quaternionic modular forms for $SU(2,n)$ were first studied by Hilado-McGlade-Yan in [HMY24], following work of Koseki-Oda ([KO95]) on $SU(2,1)$ and Yamashita ([Yam91]) on $SU(2,2)$.

Define $\tilde{D}_H$ analogously to $\tilde{D}_G$, and let $pr_-^H : \mathbb{V}_{n+\frac{\epsilon}{2},m}^\vee \otimes \mathfrak{p}_H^\vee \to \mathrm{Sym}^{2n+\epsilon-1}(V_2^{long})^\vee \boxtimes (V_H^\vee \otimes \det^{-m})$, and set $\mathcal{D}_{n+\frac{\epsilon}{2},m}^H = pr_-^H \circ \tilde{D}_H$. Then

Definition 3.2.2. Let $n \geq 1$ be an integer, $\epsilon \in \{0,1\}$, and m an integer with the same parity as ϵ. Let $\varphi : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{V}_{n+\frac{\epsilon}{2},m}^\vee$. We say that φ is a modular form on H of weight $(n+\frac{\epsilon}{2}, m)$ if

(1) $\varphi(hk) = k^{-1}\varphi(h)$ for all $h \in H(\mathbb{A})$ and $h \in K_H$
(2) $\mathcal{D}_{n+\frac{\epsilon}{2},m}^H\varphi \equiv 0$
(3) φ is smooth, moderate growth, and $Z(\mathfrak{h})$-finite.

3.3. **Fourier Expansion.** Let Φ be a QMF on G of weight n. For $g = g_f g_\infty \in G(\mathbb{A})$ with $g_f \in G(\mathbb{A}_f)$ and $g_\infty \in G(\mathbb{R})$, The Fourier expansion of Φ takes the form

$$\Phi_Z(g_f g_\infty) = \Phi_0(g_f g_\infty) + \sum_{\substack{\omega \in 2\pi W_G(\mathbb{Q}) \\ \omega > 0}} a_\omega(g_f) W_{\omega,n}^G(g_\infty)$$

where, for $\omega \in W_G(\mathbb{R})$, and $m \in M_G(\mathbb{R})$,

$$W_{\omega,n}^G(m) = \sum_{-n \leq v \leq n} W_{\omega,n}^{G,v}(m)[x^{n+v}][y^{n-v}]$$

with

$$W_{\omega,n}^{G,v}(m) = \left(\frac{|j(m,i)p_\omega(z)|}{j(m,i)p_\omega(z)}\right)^v \det(m)^n |\det(m)| K_v(|j(m,i)p_\omega(z)|).$$

and $W_{\omega,n}^G$ is determined by this formula along with equivariance properties

$$W_{\omega,n}^G(nmk) = e^{-i\langle\omega,n\rangle} k^{-1} W_{\omega,n}^G(m)$$

for $n \in N_H(\mathbb{R}), k \in K_G$. In this formula, the K_v are K-Bessel functions or modified Bessel functions of the second kind, at integer parameters v. More about the components of this general formula can be found in [Pol21] or [Pol20]. For our purposes, in Section 4, we will work with these functions in a specific choice of coordinates, so we do not go into more detail here.

Let φ be a QMF on H of weight $(n+\frac{\epsilon}{2}, m)$. The Fourier expansion of φ takes the form

$$\varphi_Z(h_f h_\infty) = \varphi_0(h_f h_\infty) + \sum_{\substack{\mu \in 2\pi W_H(\mathbb{Q}) \\ \mu \neq (0,0)}} a_\mu(h_f) W_{\mu,n+\frac{\epsilon}{2}}^H(h_\infty).$$

Here, for $\lambda \in A_H(\mathbb{R}) \cong \mathbb{R}^\times$,

$$W^H_{\mu,n+\frac{\epsilon}{2},m}(\lambda) = \sum_{\substack{-n-\frac{\epsilon}{2}\le v\le n+\frac{\epsilon}{2} \\ v\in\mathbb{Z}+\frac{\epsilon}{2}}} W^{H,v}_{\mu,n+\frac{\epsilon}{2},m}(\lambda)\cdot [x^{n+v}][y^{n-v}] \boxtimes w_{-m}$$

with

$$W^{H,v}_{\mu,n+\frac{\epsilon}{2},m}(\lambda) = \lambda^{2n+\epsilon+2}\left(\frac{|p_\mu(i)|}{p_\mu(i)}\right)^{v+\frac{m}{2}} K_{v+\frac{m}{2}}(|p_\mu(i)|\lambda)$$

The w_{-m} is just meant to record that $W^H_{\mu,n+\frac{\epsilon}{2}}$ is valued in the K_H representation $(\mathrm{Sym}^{2n+\epsilon}(V_2)\boxtimes \det^m)^\vee \cong \mathrm{Sym}^{2n+\epsilon}(V_2)\boxtimes\det^{-m}$.

4. Differential Operators

4.1. **Overview and Strategy.**

Let $n \geq 1$, and consider the representation $\pi^G_n|_H$. Recall from our discussion in Section 3.1.2 that for any $r \geq 0$, and $m \in \{-r, -r+2, \ldots, r-2, r\}$, there is a single $\pi^H_{n+\frac{r}{2},m} \subseteq \pi^G_n|_H$ whose lowest K_H-type is cut out by some $L^n_{r,m}$ with distinguished basis element $\ell^n_{r,m}$. Let $Proj^n_{r,m}$ be the K_H-equivariant projection map

$$Proj^n_{r,m} : \mathrm{Sym}^{2n+r}(V_2^{long})\boxtimes \mathrm{Sym}^r(V_G) \to \mathrm{Sym}^{2n+r}(V_2^{long})\boxtimes L^n_{r,m},$$

unique up to scalar multiple. We pin down $Proj^n_{r,m}$ by enforcing that it is the identity on elements of the form $v\boxtimes \ell^n_{r,m}$. We will also abuse notation and write $Proj^n_{r,m}$ for the projection map $\mathrm{Sym}^{2n+r}(V_G)\to L^n_{r,m}$. The differential operator of Theorem 1.0.2 will be a suitable normalization of $(Proj^n_{r,m})^\vee \circ \tilde{D}^r$.

We need to understand the action of this operator on Fourier coefficients. Let $\omega \in 2\pi W_G(\mathbb{Q})$, and recall the definitions of z_ω and b_ω from section 2.3. Let

$$v_\omega := z_\omega(y_s^3) - b_\omega(x_s y_s^2) - \overline{b_\omega}(x_s^2 y_s) + \overline{z_\omega}(x_s)^3 \in V_G = \mathrm{Sym}^3(V_2^{short}),$$

and define

$$Q^n_{r,m}(\omega) := \frac{1}{r!}\cdot\langle \ell^n_{r,m}, v^r_\omega\rangle$$

Theorem 1.0.2 will follow from the more precise result:

Theorem 4.1.1.

$$((Proj^n_{r,m})^\vee\circ\tilde{D}^r)W^G_{\omega,n}|_H = Q^n_{r,m}(\omega)W^H_{pr(\omega),n+\frac{r}{2}}$$

The path to our proof for Theorem 4.1.1 is to explicitly find a scalar-valued version of our differential operator for the "highest weight" Whittaker functions. Recall

$$W^{G,-n}_{\omega,n} := \langle W^G_{\omega,n}, x^{2n}\rangle.$$

Theorem 4.1.2. *There exists $D^n_{r,m}\in U(\mathfrak{g})$ such that $D^n_{r,m}x^{2n}\in x^{2n+r}\boxtimes L^n_{r,m}$, and*

$$D^n_{r,m}(W^{G,-n}_{\omega,n})|_H = Q^n_{r,m}(\omega)\cdot W^{H,-(n+\frac{r}{2})}_{\omega,n+\frac{r}{2},m}$$

when both sides are restricted to $A_H(\mathbb{R})^0$.

That Theorem 4.1.1 follows from Theorem 4.1.2, as well as the proof of Theorem 4.1.2, is explained in section 5. The remainder of this section is dedicated to defining our $D^n_{r,m}$, and establishing their key properties in Theorems 4.3.3 and 4.4.8.

4.2. **Operators for highest weight - definition.** We first define the $D^n_{r,m}$. For any $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$, these are elements of $U(\mathfrak{g})$ that are meant to take $x^{2n} \in \mathbb{V}^G_n \subseteq \pi^G_n$ to a highest weight vector in the lowest-K_H type $\mathbb{V}^H_{n+\frac{r}{2},m} = \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \det^m \subseteq \pi^H_{n+\frac{r}{2},m}$ in $\pi^G_n|_H$.

Our differential operators will be defined via recursive equations that involve the following coefficients:

Definition 4.2.1. For any integers $n \geq 1, r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,

- $A^n_{r,m} := -\dfrac{1}{3}\dfrac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
- $B^n_{r,m} := -\dfrac{1}{9}\dfrac{(4n+r+m-2)(3n+r-2)(n+r-1)(r+m)}{(2n+r+m)(2n+r+m-2)(2n+r-1)(2n+r-2)}$
- $E^n_{r,m} := A^n_{r,-m}$
- $F^n_{r,m} := B^n_{r,-m}$
- $U^n_{r,m} := \dfrac{1}{2}\dfrac{(4n+r+m-2)(r+m)}{(2n+r+m-2)}$
- $V^n_{r,m} := \dfrac{1}{3}\dfrac{(4n+r-m-2)(3n+r-1)(n+r)(r-m)}{(2n+r-m)(2n+r-m-2)(2n+r-1)}$
- $S^n_{r,m} := U^n_{r,-m}$
- $T^n_{r,m} := V^n_{r,-m}$

Remark 4.2.2. It would be fair to complain that, for example, $A^n_{r,m}$ is not always well defined. However, $A^n_{r,m}$ is only really defined and needed when $-r \leq m \leq r-4$. Any other problems similarly disappear.

Definition 4.2.3. Let $D^n_{0,0} = 1, D^n_{1,1} = h_1, D^n_{1,-1} = h_{-1}$. Define inductively

$$D^n_{r,r} = h_1 D^n_{r-1,r-1} + E^n_{r,r} h_3 D^n_{r-1,r-3}$$

and

$$D^n_{r,m} = h_{-1} D^n_{r-1,m+1} + A^n_{r,m} h_{-3} D^n_{r-1,m+3} + B^n_{r,m} h_{-3} h_3 D^n_{r-2,m}$$

for $m \in \{-r, -r+2, \ldots, r-2\}$.

4.3. **Operators for highest weight - algebraic properties.** We only really care about what $D^n_{r,m}$ does to the highest weight vector $x^{2n} \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1} \subseteq \pi^G_n$. Let us make this formal.

Definition 4.3.1. Define $\mathfrak{g}_+ = \mathrm{Span}_{\mathbb{C}}\{h_{-3}, h_{-1}, h_1, h_3, e_u\}$. This is a Lie subalgebra of $\mathfrak{g}$. Let $J \subseteq U(\mathfrak{g}_+)$ be the left ideal generated by e_u. Since $[e_u, h_j] = 0$ for $j = \pm 1, \pm 3$, it is actually a two-sided ideal.

Remark 4.3.2. Note that $[\mathfrak{g}_+, \mathfrak{g}_+] = \mathbb{C}e_u$. So by the Poincaré-Birkhoff-Witt theorem, each $D^n_{r,m}$ when taken modulo J is represented by a homogeneous degree r polynomial in the h_i. Furthermore, by our discussion in 3.1.2, no nonzero homogeneous degree r polynomial in the h_i can annihilate x^{2n}.

From now on, we identify $D^n_{r,m}$ with its image in $U(\mathfrak{g}_+)/J$, in other words with a unique polynomial in $\mathbb{Q}[h_{-3}, h_{-1}, h_1, h_3]$.

From the shape of the recurrence formulas, one might guess that there are some symmetries in the formulas defining $D^n_{r,m}$. We will capture some of these with the element $w_s \in K_G$ defined by

$$w_s(x) = x \qquad w_s(y) = y$$
$$w_s(x_s) = iy_s \qquad w_s(y_s) = ix_s$$

For the next theorem statement to make sense, we must note that $Jx^{2n} = 0$, $Ad(w_s)J \subseteq J$, and $[\mathfrak{k}, J] \subseteq J$.

Theorem 4.3.3 (Algebraic Properties of $D^n_{r,m}$)**.**
Let $n \geq 1$. Recall (section 3.1) the (limit of) discrete series representation π^G_n has lowest K_G-type $\mathbb{V}_n = \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. Let $x^{2n} \in \mathbb{V}_n$ be a highest weight vector. For any $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,

- *(I) $D^n_{r,m}x^{2n}$ is an Ω_H-eigenvector with eigenvalue $\lambda^n_{r,m}$, and $D^n_{r,m}x^{2n} \in x^{2n+r} \boxtimes L^n_{r,m}$ where $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^n_{r,m} \subseteq \pi^G_n$ is the lowest K_H-type of the unique $\pi^H_{n+\frac{r}{2},m} \subseteq \pi^G_n|_H$.*
- *(II) $i^r Ad(w_s)D^n_{r,m} = D^n_{r,-m}$*
- *(III) $D^n_{r,m} = h_1 D^n_{r-1,m-1} + E^n_{r,m}h_3 D^n_{r-1,m-3} + F^n_{r,m}h_3h_{-3}D^n_{r-2,m}$ for $-r < m \leq r$*
- *(IV) $[f_s, D^n_{r,m}] = U^n_{r,m}D^n_{r,m-2} + V^n_{r,m}h_{-3}D^n_{r-1,m+1}$*
- *(V) $[e_s, D^n_{r,m}] = S^n_{r,m}D^n_{r,m+2} + T^n_{r,m}h_3 D^n_{r-1,m-1}$*

Proof. Since n is fixed throughout, we will omit it from the superscripts in our notation.

We proceed by induction on r. Assume we have all (I)-(V) for all $r' < r$.

The proof of (II) for r will require (I) for r. The proof of (III) for r requires (II) for r. The proof of (IV) for r requires (III) for r. The proof of (V) for r requires (II) and (IV) for r.

(I) When $m < r$, we use the defining recurrence $D_{r,m} = h_{-1}D_{r-1,m+1} + A_{r,m}h_{-3}D_{r-1,m+3} + B_{r,m}h_3h_{-3}D_{r-2,m}$. Since Ω_H commutes with h_3 and h_{-3}, the only thing we really need to compute is $\Omega_H h_{-1}D_{r-1,m+1}x^{2n}$.

For this, note that

$$\Omega_H h_{-1} - h_{-1}\Omega_H = -[h_{-1}, \Omega_H] = h_{-1}\left(\frac{2}{3} - \frac{1}{3}h_s + h_u\right) + 2d_{-1}e_u - \frac{2}{3}h_{-3}e_s$$

Each summand can be handled separately:

- $h_{-1}\left(\frac{2}{3} - \frac{1}{3}h_s + h_u\right)D_{r-1,m+1}x^{2n} = \left(\frac{2}{3} - \frac{m+1}{3} + 2n + r - 1\right)h_{-1}D_{r-1,m+1}x^{2n}$
- $e_u D_{r-1,m+1}x^{2n} = [e_u, D_{r-1,m+1}]x^{2n} + D_{r-1,m+1}e_u x^{2n} = 0.$
- $-\frac{2}{3}h_{-3}e_s D_{r-1,m+1}x^{2n} = -\frac{2}{3}h_{-3}\left(S_{r-1,m+1}D_{r-1,m+3} + T_{r-1,m+1}h_3 D_{r-2,m}\right)$ by Statement (IV) for $r' = r-1$.

Therefore,

$$\Omega_H h_{-1} D_{r-1,m+1} x^{2n} = h_{-1} \Omega_H D_{r-1,m+1} x^{2n} + \left(\frac{2}{3} - \frac{m+1}{3} + 2n + r - 1\right) h_{-1} D_{r-1,m+1} x^{2n}$$
$$- \frac{2}{3} S_{r-1,m+1} h_{-3} D_{r-1,m+3} - \frac{2}{3} T_{r-1,m+1} h_{-3} h_3 D_{r-2,m}$$

Now observing that

(a) $\lambda^n_{r,m} = \lambda^n_{r-1,m+1} + \left(\frac{2}{3} - \frac{m+1}{3} + 2n + r - 1\right)$
(b) $A_{r,m}\lambda^n_{r,m} = -\frac{2}{3} S_{r-1,m+1} + A_{r,m}\lambda^n_{r-1,m+3}$
(c) $B_{r,m}\lambda^n_{r,m} = -\frac{2}{3} T_{r-1,m+1} + B_{r,m}\lambda^n_{r-2,m}$

we conclude $\Omega_H D_{r,m} x^{2n} = \lambda^n_{r,m} D_{r,m} x^{2n}$.

Then, since $[h_u, D_{r,m}x^{2n}] = (2n+r) D_{r,m} x^{2n}$, and $[e_u, D_{r,m}x^{2n}] = 0$, we must have $D_{r,m}x^{2n} \in x^{2n+r} \boxtimes \mathrm{Sym}^r(V_G)$. Combining this data with its Ω_H-eigenvalue and the fact $[h_s, D_{r,m}x^{2n}] = mx^{2n}$ allows us to pin down that $D_{r,m}x^{2n} \in x^{2n+r} \boxtimes L^n_{r,m}$, as desired.

To prove the statement when $m = r$, we can use the other recurrence for $D_{r,r}$ and copy the same method.

(II) For $-r < m < r$, first of all $Ad(w_s)\Omega_H = \Omega_H$ and w_s fixes x^{2n} so

$$\begin{aligned}
\Omega_H w_s D_{r,m} w_s^{-1} x^{2n} &= Ad(w_s)\Omega_H w_s D_{r,m} w_s^{-1} x^{2n} \\
&= w_s \Omega_H D_{r,m} x^{2n} \\
&= \lambda^n_{r,m} w_s D_{r,m} x^{2n}, \text{ using Statement (I)} \\
&= \lambda^n_{r,m} Ad(w_s) D_{r,m} w_s x^{2n} \\
&= \lambda^n_{r,m} Ad(w_s) D_{r,m} x^{2n}
\end{aligned}$$

Since $Ad(w_s)h_j = (-i)h_{-j}$ for $j = \pm1, \pm3$, we also see that $[h_s, Ad(w_s)D_{r,m}] = -(-i)^r m Ad(w_s) D_{r,m}$. Furthermore $[h_u, Ad(w_s)D_{r,m}] = (-i)^r r Ad(w_s) D_{r,m}$ and $[e_u, Ad(w_s)D_{r,m}] = 0$. So $Ad(w_s)D_{r,m}x^{2n}$ must be in the space $x^{2n+r} \boxtimes L^n_{r,-m}$. Therefore, in their representation as degree r homogeneous polynomials in the h_j, $j = \pm1, \pm3$, $Ad(w_s)D_{r,m} \approx D_{r,-m}$ up to scalar multiple (see remark 4.3.2). Comparing the coefficients of $h_{-1}^{\frac{r+m}{2}} h_1^{\frac{r-m}{2}}$ in $i^r Ad(w_s) D_{r,m}$ and in $D_{r,-m}$, we see that they must be equal.

(III) Here $m > -r$. Using (II),

$$\begin{aligned}
D_{r,m} &= i^r Ad(w_s) D_{r,-m} \\
&= i^r Ad(w_s)(h_{-1} D_{r-1,-m+1} + A_{r,-m} h_{-3} D_{r-1,-m+3} + B_{r,-m} h_3 h_{-3} D_{r-2,m} \\
&= h_1 D_{r-1,m-1} + E_{r,m} h_3 D_{r-1,m-3} + F_{r,m} h_3 h_{-3} D_{r-2,m}
\end{aligned}$$

by induction.

(IV)

We first handle $m > -r$. From the formula for $D_{r,m}$ from (III) and inductive hypothesis,

$$
\begin{aligned}
[f_s, D_{r,m}] &= 2h_{-1}D_{r-1,m-1} + h_1[f_s, D_{r-1,m-1}] \\
&\quad + E_{r,m}(3h_1 D_{r-1,m-3} + h_3[f_s, D_{r-1,m-3}]) + F_{r,m}(3h_{-3}h_1 D_{r-2,m} + h_{-3}h_3[f_s, D_{r-2,m}]) \\
&= 2h_{-1}D_{r-1,m-1} + h_1(U_{r-1,m-1}D_{r-1,m-3} + V_{r-1,m-1}h_{-3}D_{r-2,m}) \\
&\quad + 3E_{r,m}h_1 D_{r-1,m-3} + E_{r,m}h_3(U_{r-1,m-3}D_{r-1,m-5} + V_{r-1,m-3}h_{-3}D_{r-2,m-2}) \\
&\quad + 3F_{r,m}h_{-3}h_1 D_{r-2,m} + F_{r,m}h_{-3}h_3(U_{r-2,m}D_{r-2,m-2} + V_{r-2,m}h_{-3}D_{r-3,m+1})
\end{aligned}
$$

We collect everything into the following groups of terms:

(a) $2h_{-1}D_{r-1,m-1} + h_{-3}((V_{r-1,m-1} + 3F_{r,m})h_1 D_{r-2,m} + F_{r,m}V_{r-2,m}h_3 h_{-3}D_{r-3,m+1})$

(b) $(U_{r-1,m-1} + 3E_{r,m})h_1 D_{r-1,m-3} + E_{r,m}U_{r-1,m-3}h_3 D_{r-1,m-5}$

(c) $(E_{r,m}V_{r-1,m-3} + F_{r,m}U_{r-2,m})h_3 h_{-3}D_{r-2,m-2}$

and in each group apply the respective identities:

(a)

$$
\begin{aligned}
V_{r-1,m-1} + 3F_{r,m} &= V_{r,m} + 2A_{r,m-2} \\
F_{r,m}V_{r-2,m} &= (V_{r,m} + 2A_{r,m-2})F_{r-1,m+1}
\end{aligned}
$$

(b)

$$E_{r,m}U_{r-1,m-3} = (U_{r-1,m-1} + 3E_{r,m})E_{r,m-2}$$

(c)

$$E_{r,m}V_{r-1,m-3} + F_{r,m}U_{r-2,m} = (V_{r,m} + 2A_{r,m-2})E_{r-1,m+1} + 2B_{r,m-2} + (U_{r-1,m-1} + 3E_{r,m})F_{r,m-2}$$

Now, the second term in (a) can be written as

$$(V_{r,m} + 2A_{r,m-2})h_{-3}(h_1 D_{r-2,m} + F_{r-1,m+1}h_{-3}h_3 D_{r-3,m+1})$$

which, after taking the $(V_{r,m} + 2A_{r,m-2})h_{-3}(E_{r-1,m+1}h_3 D_{r-2,m-2})$ from (c), becomes

$$
\begin{aligned}
&(V_{r,m} + 2A_{r,m-2})h_{-3}(h_1 D_{r-2,m} + E_{r-1,m+1}h_3 D_{r-2,m-2} + F_{r-1,m+1}h_{-3}h_3 D_{r-3,m+1}) \\
&\qquad = (V_{r,m} + 2A_{r,m-2})h_{-3}D_{r-1,m+1}.
\end{aligned}
$$

Here we used Statement (III) for $r-1$.

Finally combining the above with the first term in (a) and the $2B_{r,m-2}h_3 h_{-3}D_{r-2,m-2}$ from (c) yields

$$
\begin{aligned}
&2h_{-1}D_{r-1,m-1} + 2A_{r,m-2}h_{-3}D_{r-1,m+1} + 2B_{r,m-2}h_3 h_{-3}D_{r-2,m-2} + V_{r,m}h_{-3}D_{r-1,m+1} \\
&\qquad = 2D_{r,m-2} + V_{r,m}h_{-3}D_{r-1,m+1}.
\end{aligned}
$$

Next combining (b) with the remaining $(U_{r-1,m-1}+3E_{r,m})F_{r,m-2}h_3 h_{-3}D_{r-2,m-2}$ in (c) yields, after applying Statement (III) for r,

$$
\begin{aligned}
&(U_{r-1,m-1} + 3E_{r,m})(h_1 D_{r-1,m-3} + E_{r,m-2}h_3 D_{r-1,m-5} + F_{r,m-2}h_3 h_{-3}D_{r-2,m-2}) \\
&\qquad = (U_{r-1,m-1} + 3E_{r,m})D_{r,m-2}.
\end{aligned}
$$

To finish, we observe that $(U_{r-1,m-1} + 3E_{r,m}) + 2 = U_{r,m}$.

When $m = -r$,

$$\begin{aligned}[f_s, D_{r,-r}] = h_{-3}D_{r-1,-r+1} + h_{-1}[f_s, D_{r-1,-r+1}] + A_{r,-r}h_{-3}[f_s, D_{r-1,-r+3}] \\ = h_{-3}D_{r-1,-r+1} + h_{-1}(V_{r-1,-r+1}h_{-3}D_{r-2,-r+2}) \\ + A_{r,-r}h_{-3}(U_{r-1,-r+3}D_{r-1,-r+1} + V_{r-1,-r+3}h_{-3}D_{r-2,-r+4})\end{aligned}$$

Which we group into:

(a) $h_{-3}(V_{r-1,-r+1}h_{-1}D_{r-2,-r+2} + A_{r,-r}V_{r-1,-r+3}h_{-3}D_{r-2,-r+4})$
(b) $(1 + A_{r,-r}U_{r-1,-r+3})h_{-3}D_{r-1,-r+1}$

In (a), we use the identity

$$A_{r,-r}V_{r-1,-r+3} = V_{r-1,-r+1}A_{r-1,-r+1}$$

to get

$$h_{-3}V_{r-1,-r+1}(h_{-1}D_{r-2,-r+2} + A_{r-1,-r+1}D_{r-2,-r+4}) = V_{r-1,-r+1}h_{-3}D_{r-1,-r+1}$$

When we add this to (b), we get

$$(1 + A_{r,-r}U_{r-1,-r+3} + V_{r-1,-r+1})h_{-3}D_{r-1,-r+1} = V_{r,-r}h_{-3}D_{r-1,-r+1}$$

as desired (note $U_{r,-r} = 0$).

(V) We use (II) and (IV) for r. Note that $Ad(w_s)e_s = f_s$. So

$$\begin{aligned}[e_s, D_{r,m}] &= [Ad(w_s)f_s, i^r Ad(w_s)D_{r,-m}] \\ &= i^r Ad(w_s)[f_s, D_{r,-m}] \\ &= i^r Ad(w_s)(U_{r,-m}D_{r,-m-2} + h_{-3}V_{r,-m}D_{r-1,-m+1}) \\ &= U_{r,-m}D_{r,m+2} + V_{r,-m}h_{-3}D_{r-1,m-1}\end{aligned}$$

as desired. □

4.4. **Operators for highest weight - analytic properties.** The focus of this section is to prove Theorem 4.1.2. This will follow from our more comprehensive Theorem 4.4.8. Throughout this section, we will work with a fixed $\omega \in 2\pi W_G(\mathbb{Q})$.

4.4.1. *Coordinates.* We will work with coordinates for our Whittaker functions, following the notation and results of [Pol21]. Recall $M_G(\mathbb{R}) \cong GL_2(\mathbb{R})$. Let $B(\mathbb{R})^0$ be the neutral component of the Borel subgroup of $M_G(\mathbb{R})$. Take as coordinates for $B(\mathbb{R})^0$

$$(x, y, t) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \text{ where } x \in \mathbb{R}, y \in \mathbb{R}^\times_{>0}, t \in \mathbb{R}^\times_{>0}.$$

Also write $z = x + iy$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

For some character $\omega \in 2\pi W_G(\mathbb{Q})$, and weight n, the Whittaker functions for G and their components $W^{G,v}_{\omega,n}$ are determined by their restriction to $B(\mathbb{R})^0$, because of their left N_G-equivariance and right K_G-equivariance properties. On these coordinates,

$$W^{G,v}_{\omega,n}(x,y,t) = t^{2n+2}\left(\frac{|p_\omega(z)|}{p_\omega(z)}\right)^v K_v(|p_\omega(z)|y^{-3/2}t).$$

On the other hand, recall that $M_H(\mathbb{R}) = S_H(\mathbb{R})A_H(\mathbb{R}) \cong U(1)\times\mathbb{R}^\times$ in its decomposition into a compact and split part. We take $t\in\mathbb{R}^\times_{>0}$ as a coordinate for $A_H(\mathbb{R})^0$. Then the Whittaker functions for H are determined by their restriction to $A_H(\mathbb{R})^0$. For $\mu\in 2\pi W_H(\mathbb{Q})$, and weight $(n+\frac{\epsilon}{2},m)$,

$$W^{H,v}_{\mu,n+\frac{\epsilon}{2},m}(t) = t^{2n+\epsilon+2}\left(\frac{|p_\mu(i)|}{p_\mu(i)}\right)^{v+\frac{m}{2}} K_{v+\frac{m}{2}}(|p_\mu(i)|t).$$

In our embedding of H inside G, the neutral component $A_H(\mathbb{R})^0 = M_H(\mathbb{R})\cap B(\mathbb{R})^0 = \left\{\begin{pmatrix} t & 0\\ 0 & t\end{pmatrix}, t\in\mathbb{R}^\times_{>0}\right\}$. Suppose that $\epsilon=0$, $m=0$, and $\mu = pr(\omega)$ or in other words $\chi_\mu = \chi|_{N_H}$; Then we have $W^{H,v}_{\mu,n}(t) = W^{G,v}_{\omega,n}(x,y,t)|_{A_H(\mathbb{R})} = W^{G,v}_{\omega,n}(x,y,t)|_{x=0,y=1}$.

On these coordinates, we have explicit formulas for the actions of $\mathfrak{p}$. First of all, by right K_G-equivariance, for any $m\in M_G(\mathbb{R})$

- $h_u W^{G,v}_{\omega,n}(m) = -2vW^{G,v}_{\omega,n}(m)$ for any $-n\le v\le n$
- $k_s W^{G,v}_{\omega,n}(m) = 0$ for all $k_s\in\{e_s,h_s,f_s\}$ for any $-n\le v\le n$
- $e_u W^{G,-n}_{\omega,n}(m) = 0$

Also, since $W^G_{\omega,n}(nm) = e^{-i\langle\omega,n\rangle}W^{G,v}_{\omega,n}(m)$, it follows that for any $X\equiv(a',\frac{b'}{3},\frac{c'}{3},d')\in N_G/[N_G,N_G]$ and any $m\in M_G(\mathbb{R})$,

$$(XW^G_{\omega,n})(m) = -i\langle\omega, m\cdot p_X\rangle W^{G,v}_{\omega,n}(m)$$

Next, following [Pol21], we have the following Iwasawa decompositions (modulo $\mathbb{C}E_{13}$, which acts trivially on anything in $U(\mathfrak{g})\cdot W^G_{\omega,n}$):

- $h_3 = p_3 + k_3 := -2(\epsilon_1+\epsilon_2) - (h_u+h_r)$
- $h_1 = p_1 + k_1 := 2(v+iu)^2(v-iu) - \frac{4}{3}f_r$
- $h_{-1} = p_{-1}+k_{-1} := \frac{2}{3}(\epsilon_1-\epsilon_2-2iv_2) + \frac{1}{3}(3h_u - h_s)$
- $h_{-3} = p_{-3}+k_{-3} := -2(v+iu)^3 - 4e_u$

One can calculate the effect of these differential operators on the $W^{G,v}_{\omega,n}(x,y,t)$. Recall that, for any $\phi(x,y,t): B(\mathbb{R})^0\to\mathbb{C}$ smooth,

- $(\epsilon_1+\epsilon_2)\phi = w\partial_w\phi$
- $(\epsilon_1-\epsilon_2)\phi = 2y\partial_y\phi$
- $v_2\phi = y\partial_x\phi$

First define the following auxiliary functions on $P_G(\mathbb{R})$. These come out of the derivatives of $W^{G,v}_{\omega,n}(nm) = e^{-i\langle\omega,n\rangle}W^{G,v}_{\omega,n}(m)$ by $p_1, p_{-3}\in\mathrm{Lie}(N_G)$ or their conjugates.

Definition 4.4.1.

- $Z^\omega_1(nm) := -i\langle\omega, m\cdot p_1\rangle$
- $Z^\omega_{-1}(nm) := -i\langle\omega, m\cdot -\overline{p_1}\rangle$
- $Z^\omega_{-3}(nm) := -i\langle\omega, m\cdot\overline{p_{-3}}\rangle$.
- $Z^\omega_3(nm) := -i\langle\omega, m\cdot -p_{-3}\rangle$.

Note that these Z_j, when restricted to $m=(x,y,t)\in B(\mathbb{R})^0$, do not depend on t.

Lemma 4.4.2. *On $B(\mathbb{R})^0$,*

- $Z^{\omega}_{-3}(z) := -2y^{-3/2}p_\omega(z^*)$
- $Z^{\omega}_{-1}(z) := \frac{4}{3}iy^{5/2}\partial_{z^*}(p_\omega(z^*)y^{-3})$
- $Z^{\omega}_{1}(z) := \frac{4}{3}iy^{5/2}\partial_{z}(p_\omega(z)y^{-3})$
- $Z^{\omega}_{3}(z) := 2y^{-3/2}p_\omega(z)$

Proof. This is [Pol21] Lemma 4.5 with some rescaling. $\square$

These appear in the derivatives of our Whittaker functions. Define functions $G^\omega_v : B(\mathbb{R})^0 \to \mathbb{C}$ by $W^{G,v}_{\omega,n}(x,y,t) = t^{2n+2}G^\omega_v(x,y,t)$, i.e.

$$G^\omega_v(x,y,t) = \left(\frac{|p_\omega(z)|}{p_\omega(z)}\right)^v K_v(|p_\omega(z)|y^{-3/2}t)$$

Lemma 4.4.3. *On $B(\mathbb{R})^0$, $p_{-1} = -\frac{8}{3}iy\partial_{\bar z}$ and $p_3 = -2t\partial_t$, and*

(1) $-\frac{8}{3}iy\partial_{\bar z}G^\omega_v = Z^\omega_{-1}tG^\omega_{v-1} + 2vG^\omega_v$
(2) $-2t\partial_t G^\omega_v = -Z^\omega_{-3}tG^\omega_{v-1} + 2vG^\omega_v$
(3) $-2t\partial_t G^\omega_v = Z^\omega_{3}tG^\omega_{v+1} - 2vG^\omega_v$
(4) $-Z^\omega_3 tG^\omega_v = Z^\omega_{-3}tG^\omega_{v-2} - 4(v-1)G^\omega_{v-1}$

Proof. 1,2, and 3 are properties of K-Bessel functions; see section 4 of [Pol21]. 4 follows immediately from 2 and 3. $\square$

Lemma 4.4.4.

(1) $p_1 XW^{-n}_{\omega,n}(m) = Z^\omega_1(m)t \cdot XW^{-n}_{\omega,n}(m)$
(2) $p_{-3} XW^{-n}_{\omega,n}(m) = Z^\omega_3(m)t \cdot XW^{-n}_{\omega,n}(m)$

for all $X \in U(\mathfrak{g})$ and $m \in B(\mathbb{R})^0$.

Proof. Follows from left N_G-equivariance. $\square$

Finally we will need the relations

Lemma 4.4.5. *On $B(\mathbb{R})^0$,*

- $p_{-1}(Z^\omega_{-3}) = 2Z^\omega_{-3} + 4Z^\omega_{-1}$
- $p_{-1}(Z^\omega_{-1}) = \frac{2}{3}Z^\omega_{-1} + \frac{8}{3}Z^\omega_{1}$
- $p_{-1}(Z^\omega_{1}) = -\frac{2}{3}Z^\omega_{1} + \frac{4}{3}Z^\omega_{3}$
- $p_{-1}(Z^\omega_{3}) = -2Z^\omega_{3}$

Proof. Direct computation. $\square$

Definition 4.4.6.

Define $P^{n,\omega}_{0,0} = 1, P^{n,\omega}_{1,1} = Z^\omega_1, P^{n,\omega}_{1,-1} = Z^\omega_{-1}$. Then, define $P^{n,\omega}_{r,m}$ by the recurrence formulas.

$$P^{n,\omega}_{r,r} = Z^\omega_1 P^{n,\omega}_{r-1,m-1} + E^n_{r,r}Z^\omega_3 P^{n,\omega}_{r-1,m-3}$$

and

$$P^{n,\omega}_{r,m} = Z^\omega_{-1}P^{n,\omega}_{r-1,m+1} + A^n_{r,m}Z^\omega_{-3}P^{n,\omega}_{r-1,m+3} + B^n_{r,m}Z^\omega_3 Z^\omega_{-3}P^{n,\omega}_{r-2,m}$$

for $m \in \{-r, -r+2, \ldots, r-2\}$.

4.4.2. *Main result.* In this section we calculate the action of $D_{r,m}$ on

$$W^{G,-n}_{\omega,n} = \langle W^G_{\omega,n}, x^{2n}\rangle.$$

Recall we are identifying $D_{r,m}$ with a homogeneous degree r polynomial in $\mathbb{Q}[h_{-3}, h_{-1}, h_1, h_3]$. So we need to make sure this action is well-defined:

Lemma 4.4.7. *Let $F : G(\mathbb{R}) \to \mathbb{V}_n^\vee$ be a smooth, right K_G-equivariant function. If $Y \in J$, then*

$$Y\langle F, x^{2n}\rangle = 0$$

Proof. We may assume Y is a monomial. By the Poincaré-Birkhoff-Witt theorem, we can write $Y = Y_1 e_u^s$ with $Y_1 \in \mathfrak{p}_+^{\otimes r}$ for some $r \geq 0$ and $s > 0$ Then,

$$Y\langle F, x^{2n}\rangle = -Y_1\langle F, e_u^s x^{2n}\rangle = 0.$$

□

We are now ready to state the main result of this section:

Theorem 4.4.8 (Analytic properties of $D_{r,m}$)**.** *Let $n \geq 1$ be an integer, and $\omega \in W_G(\mathbb{R})$. Then, for any $r \geq 0$ and $m \in \{-r, -r+2, \dots, r-2, r\}$,*

(I) On the group $B(\mathbb{R})^0$,

$$D^n_{r,m}(W^{-n}_{\omega,n}) = P^{n,\omega}_{r,m}(z)t^{2n+r+2}G_{-n-(\frac{r-m}{2})}(x,y,t)$$

(II) For $m > -r$, there is the additional recurrence formula

$$P^{n,\omega}_{r,m} = Z^\omega_1 P^{n,\omega}_{r-1,m-1} + E^n_{r,m}Z^\omega_3 P^{n,\omega}_{r-1,m-3} + F^n_{r,m}Z^\omega_3 Z^\omega_{-3}P^{n,\omega}_{r-2,m}$$

(III) Define $S_{r,m}$ for $r \geq 0$ and $m \in \{-r, -r+2, \dots, r-2\}$ by $S_{0,0} = S_{1,1} = S_{1,-1} = 0$, and for $r \geq 2$

$$S_{r,m} := \frac{2}{3}(m+1)P^{n,\omega}_{r-1,m+1} + p_{-1}(P^{n,\omega}_{r-1,m+1}) + 2A^n_{r,m}(2n+r-m-2)P^{n,\omega}_{r-1,m+3} + 4B^n_{r,m}(2n+r-1)(Z_3P^{n,\omega}_{r-2,m}).$$

Then, $S_{r,m} = 0$ for all r and m.

Proof. Since n and ω are fixed throughout, we omit them from the notation.

Outline: We proceed by induction on r. We may verify directly that the theorem holds for $r = 0$ and $r = 1$.

Suppose now that $r \geq 2$ and we have Statements (I), (II), and (III) for all $r' < r$.

We first prove (III) for r, which requires (II) and (III) for r'. Next we prove (I) for r, which will require (III) for r as well as the inductive hypothesis. Then we finally prove (II), which will require (I) for r as well as the inductive hypothesis.

(III) We handle separately the three cases $m \in \{-r, -r+2, \dots, r-6\}$ and $m = r-4$ and $m = r-2$, due to the different recurrence formulas defining $P_{r-1,m+1}$ and $P_{r-1,m+3}$ in these cases.

Case 1 of (III), $m \in \{-r, -r+2, \dots, r-6\}$:

We will spell out the computations for this case in the detail. Set $C_{r,m} = 4B_{r,m}(2n+r-1) - \frac{8}{3}E_{r-1,m+3}$. Then we can write

$$S_{r,m} = \frac{2}{3}(m+1)P_{r-1,m+1} + p_{-1}(P_{r-1,m+1}) + 2A_{r,m}(2n+r-m-2)P_{r-1,m+3} + C_{r,m}(Z_{-3}P_{r-2,m}) + \frac{8}{3}E_{r-1,m+3}(Z_3P_{r-2,m}).$$

Into the expression above, we will plug in the recurrence formulas of definition 4.4.6 for $P_{r-1,m+1}$ and $P_{r-1,m+3}$ and $P_{r-2,m}$ in for everything except the last summand.

We first compute, using the recurrence formula for $P_{r-1,m+1}$,

$$\begin{aligned} p_{-1}(P_{r-1,m+1}) &= p_{-1}(Z_{-1}P_{r-2,m+2} + A_{r-1,m+1}Z_{-3}P_{r-2,m+4} + B_{r-1,m+1}Z_3Z_{-3}P_{r-3,m+1}) \\ &= p_{-1}(Z_{-1})P_{r-2,m+2} + Z_{-1}p_{-1}(P_{r-2,m+2}) \\ &\quad + A_{r-1,m+1}\left[p_{-1}(Z_{-3})P_{r-2,m+4} + Z_{-3}p_{-1}(P_{r-2,m+4})\right] \\ &\quad + B_{r-1,m+1}\left[p_{-1}(Z_{-3}Z_3)P_{r-3,m+1} + Z_{-3}Z_3p_{-1}(P_{r-3,m+1})\right] \end{aligned}$$

which using Lemma 4.4.5 gives

$$\begin{aligned} p_{-1}(P_{r-1,m+1}) &= \frac{2}{3}Z_{-1}P_{r-2,m+2} + \frac{8}{3}Z_1P_{r-2,m+2} + Z_{-1}p_{-1}(P_{r-2,m+2}) \\ &\quad + A_{r-1,m+1}\left[2Z_{-3}P_{r-2,m+4} + 4Z_{-1}P_{r-2,m+4} + Z_{-3}p_{-1}(P_{r-2,m+4})\right] \\ &\quad + B_{r-1,m+1}\left[2Z_{-3}Z_3P_{r-3,m+1} + 4Z_{-1}Z_3P_{r-3,m+1} - 2Z_{-3}Z_3P_{r-3,m+1} + Z_{-3}Z_3p_{-1}(P_{r-3,m+1})\right] \end{aligned}$$

Now plugging in the above, as well as the recurrence formulas for

- $P_{r-1,m+1} = Z_{-1}P_{r-2,m+2} + A_{r-1,m+1}Z_{-3}P_{r-2,m+4} + B_{r-1,m+1}Z_3Z_{-3}P_{r-3,m+1}$
- $P_{r-1,m+3} = Z_{-1}P_{r-2,m+4} + A_{r-1,m+3}Z_{-3}P_{r-2,m+6} + B_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3}$
- $P_{r-2,m} = Z_{-1}P_{r-3,m+1} + A_{r-2,m}Z_{-3}P_{r-2,m+2} + B_{r-2,m}Z_3Z_{-3}P_{r-4,m}$

into $S_{r,m}$ (but keeping the $\frac{8}{3}E_{r-1,m+3}Z_3P_{r-2,m}$ part as is), one is left with the sum of the following groups of terms:

(a) $(2A_{r,m}(2n+r-m-2)B_{r-1,m+3} + C_{r,m}A_{r-2,m})Z_3Z_{-3}P_{r-3,m+3}$
(b) $(4A_{r-1,m+1} + 2A_{r,m}(2n+r-m-2))Z_{-1}P_{r-2,m+4}$
(c) $2A_{r,m}(2n+r-m-2)A_{r-1,m+3}Z_{-3}P_{r-2,m+6}$
(d) $\frac{2}{3}(m+2)Z_{-1}P_{r-2,m+2} + Z_{-1}p_{-1}(P_{r-2,m}+2) + (4B_{r-1,m+1} + C_{r,m})Z_{-1}Z_3P_{r-3,m+1}$
(e)

$$\frac{2}{3}(m+1)B_{r-1,m+1}Z_{-3}Z_3P_{r-3,m+1} + B_{r-1,m+1}Z_{-3}Z_3p_{-1}(P_{r-3,m+1}) + C_{r,m}B_{r-2,m}Z_{-3}Z_3Z_3P_{r-4,m}$$

(f) $\frac{2}{3}(m+4)A_{r-1,m+1}Z_{-3}P_{r-2,m+4} + A_{r-1,m+1}Z_{-3}p_{-1}(P_{r-2,m+4})$
(g) $\frac{8}{3}Z_1P_{r-2,m+2} + \frac{8}{3}E_{r-1,m+3}Z_3P_{r-2,m}$

In each group, we notice the following identities involving the coefficient terms:

(a)

$$2A_{r,m}(2n+r-m-2)B_{r-1,m+3}+C_{r,m}A_{r-2,m}=\frac{8}{3}F_{r-1,m+3}-\frac{8}{3}B_{r-1,m+3} + A_{r-1,m+1}\cdot 4B_{r-1,m+3}(2n+(r-1)-1) + B_{r-1,m+1}\cdot 2A_{r-2,m}(2n+(r-2)-m-2)$$

(b) $4A_{r-1,m+1}+2A_{r,m}(2n+r-m-2)=-\frac{8}{3}+2A_{r-1,m+1}(2n+(r-1)-(m+1)-2)$
(c) $2A_{r,m}(2n+r-m-2)A_{r-1,m+3}=-\frac{8}{3}A_{r-1,m+3}+A_{r-1,m+1}\cdot 2A_{r-1,m+3}(2n+(r-1)-(m+3)-2)$
(d) $4B_{r-1,m+1}+C_{r,m}=4B_{r-1,m+1}(2n+(r-1)-1)$
(e) $C_{r,m}B_{r-2,m}=B_{r-1,m+1}\cdot 4B_{r-2,m}(2n+(r-2)-1)$
(f) leave as is
(g) leave as is

Now,

- The $\frac{8}{3}F_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3}$ from (a), added to (g), gives
$$\frac{8}{3}Z_1P_{r-2,m+2}+\frac{8}{3}E_{r-1,m+3}Z_3P_{r-2,m}+\frac{8}{3}F_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3},$$
which is exactly $\frac{8}{3}P_{r-1,m+3}$ by (II) for $r-1$.
- The $-\frac{8}{3}B_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3}$ from (a), added to the $-\frac{8}{3}Z_{-1}P_{r-2,m+4}$ from (b), and the $-\frac{8}{3}A_{r-1,m+3}P_{r-2,m+6}$ from (c), gives $-\frac{8}{3}P_{r-1,m+3}$ by the recurrence formula defining $P_{r-1,m+3}$.
- All the terms in (d), added to the $2A_{r-1,m+1}(2n+(r-1)-(m+1)-2)Z_{-1}P_{r-2,m+4}$ from (b), combine to give $Z_{-1}\cdot S_{r-1,m+1}$, which vanishes by inductive hypothesis.
- All the terms in (e), added to the $B_{r-1,m+1}\cdot 2A_{r-2,m}(2n+(r-2)-m-2)Z_3Z_{-3}P_{r-3,m+3}$ from (a), combined to give $B_{r-1,m+1}Z_3Z_{-3}\cdot S_{r-2,m}$,which vanishes by inductive hypothesis.
- All the terms in (f), added to the $A_{r-1,m+1}\cdot 4B_{r-1,m+3}(2n+(r-1)-1)Z_3Z_{-3}P_{r-3,m+3}$ from (a), added to the $A_{r-1,m+1}\cdot 2A_{r-1,m+3}(2n+(r-1)-(m+3)-2)Z_{-3}P_{r-2,m+6}$ from (c), combine to give $A_{r-1,m+1}Z_{-3}\cdot S_{r-1,m+3}$, which vanishes by inductive hypothesis.

All terms are accounted for in the list above, and we are left with $S_{r,m}=\frac{8}{3}P_{r-1,m+3}-\frac{8}{3}P_{r-1,m+3}=0$, as desired.

Case 2 of (III), $m=r-2$:

We wish to show

$$S_{r,r-2}=\frac{2}{3}(r-1)P_{r-1,r-1}+p_{-1}(P_{r-1,r-1})+4B_{r,r-2}(2n+r-1)(Z_3P_{r-2,r-2})=0.$$

Set

$$C=4B_{r,r-2}(2n+r-1)-4B_{r-1,r-3}(2n+r-2),$$

so that

$$S_{r,r-2}=\frac{2}{3}(r-1)P_{r-1,r-1}+p_{-1}(P_{r-1,r-1})+4B_{r-1,r-3}(2n+r-2)Z_3P_{r-2,r-2}+CZ_3P_{r-2,r-2}.$$

We will plug in the recurrence formula of 4.4.6 for $P_{r-1,r-1}$ and $P_{r-2,r-2}$ in everything but the last summand above. For example, using the recurrence formula

$$P_{r-1,r-1} = Z_1 P_{r-2,r-2} + E_{r-1,r-1} Z_3 P_{r-2,r-4},$$

we compute with Lemma 4.4.5

$$\begin{aligned} p_{-1}(P_{r-1,r-1}) = -\frac{2}{3} Z_1 P_{r-2,r-2} + \frac{4}{3} Z_3 P_{r-2,r-2} + Z_1 p_{-1}(P_{r-2,r-2}) \\ - 2Z_3 E_{r-1,r-1} P_{r-2,r-4} + Z_3 E_{r-1,r-1} p_{-1}(P_{r-2,r-4}). \end{aligned}$$

In the end, $S_{r,r-2}$ is the sum of the following groups of terms:

(a)
$$\frac{2}{3}(r-2) Z_1 P_{r-2,r-2} + Z_1 p_{-1}(P_{r-2,r-2}) + 4B_{r-1,r-3}(2n+r-2) Z_1 P_{r-3,r-3}$$

(b)
$$\begin{aligned} \frac{2}{3}(r-4) Z_3 E_{r-1,r-1} P_{r-2,r-4} + Z_3 E_{r-1,r-1} p_{-1}(P_{r-2,r-4}) \\ + (\frac{4}{3} + C) Z_3 P_{r-2,r-2} + 4B_{r-1,r-3}(2n+r-2) E_{r-2,r-2} Z_3 Z_3 P_{r-3,r-5} \end{aligned}$$

Each group vanishes:

(a) This is $Z_1 S_{r-1,r-3}$, which vanishes by inductive hypothesis.

(b) We must notice that

$$B_{r-1,r-3} E_{r-2,r-2} = B_{r-1,r-5} E_{r-1,r-1}$$

and
$$\frac{4}{3} + C = E_{r-1,r-1} \cdot 2A_{r-1,r-5}(2n + (r-1) - (r-5) - 2)$$
to see that these terms can be expressed as

$$Z_3 E_{r-1,r-1} \cdot S_{r-1,r-5}$$

which again vanishes by inductive hypothesis.

Therefore we have $S_{r,r-2} = 0$, as desired.

Case 3 of (III), $m = r - 4$:

Note that
$$2A_{r,r-4}(2n + r - (r-4) - 2) = -\frac{8}{3},$$
so that

$$S_{r,r-4} = \frac{2}{3}(r-3) P_{r-1,r-3} + p_{-1}(P_{r-1,r-3}) - \frac{8}{3} P_{r-1,r-1} + 4B_{r,r-4}(2n+r-1) Z_3 P_{r-2,r-4}.$$

Set
$$C' = 4B_{r,r-4}(2n+r-1) - \frac{8}{3} E_{r-1,r-1}$$

so that

$$S_{r,r-4} = \frac{2}{3}(r-3)P_{r-1,r-3} + p_{-1}(P_{r-1,r-3}) - \frac{8}{3}P_{r-1,r-1} + C'Z_3P_{r-2,r-4} + \frac{8}{3}E_{r-1,r-1}Z_3P_{r-2,r-4}$$

We plug in the recurrence formulas of definition 4.4.6 for $P_{r-1,r-3}$ and $P_{r-1,r-1}$ and $P_{r-2,r-4}$ into everything but the last summand above; note a part of $p_{-1}(Z_{-1}P_{r-2,r-2})$ and the $\frac{8}{3}E_{r-1,r-1}Z_3P_{r-2,r-4}$ will cancel the $-\frac{8}{3}P_{r-1,r-1}$. In the end, $S_{r,r-4}$ is the sum of the following groups of terms:

(a)

$$\frac{2}{3}(r-2)Z_{-1}P_{r-2,r-2} + Z_{-1}p_{-1}(P_{r-1,r-2}) + (4B_{r-1,r-3} + C')Z_{-1}Z_3P_{r-3,r-3}$$

(b)

$$\frac{2}{3}(r-3)Z_3Z_{-3}B_{r-1,r-3}P_{r-3,r-3} + Z_3Z_{-3}B_{r-1,r-3}p_{-1}(P_{r-3,r-3}) + C'B_{r-2,r-4}Z_3Z_{-3}Z_3P_{r-4,r-4}$$

Each group vanishes:

(a) Noting that

$$4B_{r-1,r-3} + C' = 4B_{r-1,r-3}(2n + (r-1) - 1),$$

these terms combine to give $Z_{-1} \cdot S_{r-1,r-3}$ which vanishes by inductive hypothesis.

(b) Noting that

$$C'B_{r-2,r-4} = B_{r-1,r-3} \cdot 4B_{r-2,r-4}(2n + (r-2) - 1),$$

these terms combine to give $B_{r-1,r-3}Z_3Z_{-3} \cdot S_{r-2,r-4}$, which vanishes by inductive hypothesis.

Therefore $S_{r,r-4} = 0$, as desired. **(I)** We handle separately the cases $m \in \{-r, -r+2, \dots, r-2\}$ and $m = r-2$.

Case 1 of (I), $m \in \{-r, -r+2, \dots, r-2\}$:

We compute $D_{r,m}$ via its definition

$$D_{r,m} = h_{-1}D_{r-1,m+1} + A_{r,m}h_{-3}D_{r-1,m+3} + B_{r,m}h_{-3}h_3D_{r-2,m}$$

and inductive hypothesis.

We have by Lemma 4.4.3 (1),

$$\begin{aligned} h_{-1}D_{r-1,m+1}(W^{-n}_{\omega,n}) &= p_{-1}(P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}) + k_{-1}(D_{r-1,m+1}(W^{-n}_{\omega,n})) \\ &= (Z_{-1}P_{r-1,m+1})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} + 2(-n-(\frac{r-m}{2})+1)P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1} \\ &\quad + p_{-1}(P_{r-1,m+1})t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1} \\ &\quad + (2n+(r-1)-(\frac{m+1}{3}))P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

By Lemma 4.4.4 (2) and Lemma 4.4.3 (4),

$$h_{-3}D_{r-1,m+3}(W^{-n}_{\omega,n}) = -Z_3(P_{r-1,m+3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+2})$$
$$= (-Z_3P_{r-1,m+3})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} - 4(-n-(\frac{r-m}{2})+1)P_{r-1,m+3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}.$$

By Lemma 4.4.3 (3),

$$h_3D_{r-2,m}(W^{-n}_{\omega,n}) = p_3(P_{r-2,m}t^{2n+(r-2)+2}G_{-n-(\frac{r-m}{2})+1}) + k_3(D_{r-2,m}(W^{-n}_{\omega,n}))$$
$$= -Z_{-3}P_{r-2,m}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})} - 2(2n+(r-2)+2)P_{r-2,m}t^{2n+(r-2)+2}G_{-n-(\frac{r-m}{2})+1}$$
$$+ 2(-n-(\frac{r-m}{2})+1)P_{r-2,m}t^{2n+(r-2)+1}G_{-n-(\frac{r-m}{2})+1}$$
$$- (2n+(r-2)+m)P_{r-2,m}t^{2n+(r-2)+1}G_{-n-(\frac{r-m}{2})+1}$$

so then by Lemma 4.4.4 (2)

$$(\star)\quad h_{-3}h_3D_{r-2,m}(W^{-n}_{\omega,n}) = Z_3Z_{-3}P_{r-2,m}t^{2n+r+2}G_{-n-(\frac{r-m}{2})}$$
$$+ 4Z_3P_{r-2,m}(2n+r-1)t^{2n+(r-1)+1}G_{-n-(\frac{r-m}{2})+1}$$

Collecting terms, we get

$$D_{r,m}(W^{-n}_{\omega,n}) = P_{r,m}t^{2n+r+2}G_{-n-\frac{(r-m)}{2}} + (S_{r,m})t^{2n+(r-1)+1}G_{-n-(\frac{r-m}{2})+1}$$

where $S_{r,m}$ is

$$\frac{2}{3}(m+1)P_{r-1,m+1}+p_{-1}(P_{r-1,m+1})+4A_{r,m}(n+(\frac{r-m}{2})-1)P_{r-1,m+3}+4B_{r,m}(2n+r-1)(Z_3P_{r-2,m})$$

and vanishes by Statement (III).

Case 2 of (I), $m = r$:

We use Theorem 4.3.3 (III) and compute in general

$$D_{r,m}(W^{-n}_{\omega,n}) = (h_1D_{r-1,m-1} + E_{r,m}h_3D_{r-1,m-3} + F_{r,m}h_3h_{-3}D_{r-2,m})(W^{-n}_{\omega,n})$$

From Lemma 4.4.4 (1) and the formula for $[f_s, D_{r,m}]$ from Theorem 4.3.3 and Lemma 4.4.4 (2),

$$h_1D_{r-1,m-1}(W^{-n}_{\omega,n}) = p_1(D_{r-1,m-1}W^{-n}_{\omega,n}) + k_1(D_{r-1,m-1}W^{-n}_{\omega,n})$$
$$= Z_1P_{r-1,m-1}t^{2n+r+2}G_{-n-(\frac{r-m}{2})} - \frac{4}{3}[f_s, D_{r-1,m-1}]W^{-n}_{\omega,n}$$
$$= Z_1P_{r-1,m-1}t^{2n+r+2}G_{-n+\frac{r-m}{2}} - \frac{4}{3}(U_{r-1,m-1}D_{r-1,m-3} + V_{r-1,m-1}h_{-3}D_{r-2,m})W^{-n}_{\omega,n}$$
$$= Z_1P_{r-1,m-1}t^{2n+r+2}G_{-n+\frac{r-m}{2}} - \frac{4}{3}U_{r-1,m-1}P_{r-1,m-3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})-1}$$
$$+ \frac{4}{3}V_{r-1,m-1}Z_3P_{r-2,m}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}$$

From Lemma 4.4.3 (2)

$$\begin{aligned}
h_3 D_{r-1,m-3}(W_{\omega,n}^{-n}) &= p_3(P_{r-1,m-3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})-1}) + k_3(D_{r-1,m-3}(W_{\omega,n}^{-n})) \\
&= (Z_3 P_{r-1,m-3})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} - 2(2n+(r-1)+2)P_{r-1,m-3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})-1} \\
&\quad - 2(-n-(\frac{r-m}{2})-1)P_{r-1,m-3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})-1} \\
&\quad - (2n+(r-1)+(m-3))P_{r-1,m-3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})-1}
\end{aligned}$$

Using also the formula for $h_{-3}h_3 D_{r-2,m}(W_{\omega,n}^{-n})$ from $(\star)$, we get

$$\begin{aligned}
&(h_1 D_{r-1,m-1} + E_{r,m}h_3 D_{r-1,m-3} + F_{r,m}h_3 h_{-3} D_{r-2,m})(W_{\omega,n}^{-n} \\
&\quad = (Z_1 P_{r-1,m-1} + E_{r,m}Z_3 P_{r-1,m-3} + F_{r,m}Z_3 Z_{-3}P_{r-2,m})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} \\
&\quad + (S')P_{r-1,m-3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})-1} + (S'')Z_3 P_{r-2,m}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}
\end{aligned}$$

Where

$$S' = \left(-\frac{4}{3}U_{r-1}^{m-1} + (-4n-2r-2m+4)E_{r,m}\right) = 0.$$

and

$$S'' = \left(\frac{4}{3}V_{r-1}^{m-1} + 4F_{r,m}(2n+r-1)\right) = 0$$

so we are left with only

$$(\star\star) \qquad D_{r,m}(W_{\omega,n}^{-n}) = (Z_1 P_{r-1,m-1} + E_{r,m}Z_3 P_{r-1,m-3} + F_{r,m}Z_3 Z_{-3}P_{r-2,m})t^{2n+r+2}G_{-n-(\frac{r-m}{2})}$$

When $m = r$, by definition $P_{r,r} = Z_1 P_{r-1,r-1} + E_{r,r}Z_3 P_{r-1,r-3}$ so we are done (note $F_{r,r} = 0$).

(II)

When $-r < m < r$, we have on one hand from (I) that

$$D_{r,m}(W_{\omega,n}^{-n}) = P_{r,m}t^{2n+r+2}G_{-n-(\frac{r-m}{2})}$$

and on the other hand from $(\star\star)$ that

$$D_{r,m}(W_{\omega,n}^{-n}) = (Z_1 P_{r-1,m-1} + E_{r,m}Z_3 P_{r-1,m-3} + F_{r,m}Z_3 Z_{-3}P_{r-2,m})t^{2n+r+2}G_{-n-(\frac{r-m}{2})}$$

so we are done (note that the K-Bessel functions do not vanish).

□

5. Final Steps

In this section we finally prove Theorem 4.1.2 and Theorem 4.1.1, and then Theorem 1.0.2. We start with a preliminary lemma:

Lemma 5.0.1. *Let $v \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. If $Y \in \mathfrak{p}^{\otimes r}$, then for any smooth function $F : G(\mathbb{R}) \to \mathbb{V}_n^\vee$,*

$$\langle (\tilde{D}^r)F, Yv\rangle = Y\langle F, v\rangle.$$

Proof. We can check this directly, for example on basis elements of $\mathfrak{p}$ and $\mathbb{V}_n$. □

Lemma 5.0.2. *Let $L \subseteq \mathrm{Sym}^r(V_G)$ be a line. Let*

$$Pr_L : \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G) \to \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L$$

be the K_H-equivariant projection map, unique up to scalar multiple.

Let $v \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. If $Y \in \mathfrak{p}^{\otimes r}$, and satisfies $Yv \in v' \boxtimes L$ with $v' \in \mathrm{Sym}^{2n+r}(V_2^{long})$, then for any smooth function $F : G(\mathbb{R}) \to \mathbb{V}_n$,

$$\langle (Pr_L^\vee \circ \tilde{D}^r)F, v' \rangle = Y\langle F, v\rangle.$$

up to scalar multiple

Proof. First $\tilde{D}^r F$ is valued in

$$\bigoplus_{j=0}^{r} \left(\mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G)\right)^\vee .$$

We can orthogonally decompose the top summand relative to K_H:

$$\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G) = \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L \oplus \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^\perp.$$

If $Yv \in v' \boxtimes L$, it follows that

$$\langle (Pr_{L^\perp}^\vee \circ \tilde{D}^r)F, Yv \rangle = 0,$$

and so

$$\langle \tilde{D}^r F, Yv \rangle = \langle (Pr_L^\vee \circ \tilde{D}^r)F, v' \rangle$$

and we are done via Lemma 5.0.1. □

Proof of Theorem 4.1.2. Note that $P_{r,m}^{\omega,n}$ and $D_{r,m}^n$ are defined by the same recurrence relations, and the line $L_{r,m}^n$ in terms of monomials in $\{(y_s^3), (x_s y_s^2), (x_s^2 y_s), (x_s^3)\}$ can be read off from the recurrence relations defining $D_{r,m}^n$. Furthermore,

- $Z_3^\omega(i) = z_\omega$;
- $Z_1^\omega(i) = b_\omega/3$;
- $Z_{-1}^\omega(i) = -\overline{b_\omega}/3$
- $Z_{-3}^\omega(i) = -\overline{z_\omega}$.

Therefore,

$$P_{r,m}^{\omega,n}(i) = \frac{1}{r!}\langle \ell_{r,m}^n, v_\omega^r \rangle = Q_{r,m}^n(\omega)$$

and the theorem follows from Theorem 4.4.8 (I). □

Proof of Theorem 4.1.1. Both sides of Theorem 4.1.1 are quaternionic. So to test equality we can pair with an element of $x^{2n+r} \boxtimes L_{r,m}^n$. Such an element is given by $D_{r,m}^n x^{2n}$, due to Theorem 4.3.3. Then since both sides have the same N_H and K_H equivariance properties, we need to check equality on $A_H(\mathbb{R})^0$. Then, by Lemma 5.0.2, this is exactly the content of Theorem 4.1.2. □

Proof of Theorem 1.0.2. Suppose $\omega \in 2\pi W_G(\mathbb{Q})$ has $pr(\omega) = \mu \in 2\pi W_H(\mathbb{Q})$. Then, $Q_{r,m}^n(\omega) \in (2\pi)^r \mathbb{Q}(i)$, and so by Theorem 4.1.1, we can set $\mathcal{D}_{r,m}^n = \pi^{-r}(Proj_{r,m}^n)^\vee \circ \tilde{D}^r$ to get the desired algebraicity of Fourier coefficients. □

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