Special Values of L-functions on Quaternionic Groups

Bryan Hu

October 18, 2025

Deligne's Conjecture on Critical Values of L-functions

- Motivation: Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function. For positive even integers k,

$$\zeta(k) = (-1)^{\frac{k}{2}+1} \frac{(2\pi)^k B_k}{2(k!)}.$$

- General conjecture (Deligne):

$$L(k) \in (\text{period}) \cdot \overline{\mathbb{Q}}$$

  at critical values k.
- One method to prove things about L-functions is to use integral representations and properties of Eisenstein series

A result of Shimura

- Let f and g be holomorphic modular forms with Fourier expansions

$$f = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, g = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

- Define the product L-function

$$L(s, f \times g) = \sum_{n=0}^{\infty} a_n \overline{b_n} n^{-s}$$

Theorem (Shimura)

Let f be a Hecke eigenform of weight ℓ_1 and g a holomorphic modular form of weight $\ell_2 < \ell_1$. Then, when k is an integer with $\frac{1}{2}(\ell_1 + \ell_2 - 2) < s < \ell_1$,

$$\pi^{-\ell_1} \frac{L(k, f \times g)}{\langle f, f \rangle} \in \mathbb{Q}(f)\mathbb{Q}(g)$$

A result of Shimura

- Proof of theorem: Integral representation, control of Fourier coefficients / properties of Eisenstein series, Maass-Shimura operators
- Integral representation (Rankin, Selberg):

$$\langle f(z), g(z) \cdot E_n(z,s) \rangle \approx L(s + \ell_1 - 1, f \times g),$$

  where $E_n(z,s)$ is real-analytic Eisenstein series of weight $n = \ell_1 - \ell_2$
- When $s = 0$, $E_n(z,0)$ is a holomorphic Eisenstein series of weight n. So

$$\langle f, g \cdot E_n(z,0) \rangle \approx L(\ell_1 - 1, f \times g),$$

  which implies

$$\pi^{-\ell_1} \langle f, f \rangle^{-1} L(\ell_1 - 1, f \times g) \in \mathbb{Q}(f)\mathbb{Q}(g).$$

A result of Shimura

- We have the result for the *right-most* critical value in Shimura's theorem.
- To get algebraicity results for critical values to the left of $\ell_1 - 1$, use Maass-Shimura differential operators

$$\delta_n = \frac{1}{2\pi i}\left(\frac{n}{2iy} + \frac{\partial}{\partial z}\right), \delta_n^{(r)} = \delta_{n+2r-2} \circ \cdots \circ \delta_{n+2} \circ \delta_n$$

- Then $E_{n+2r}(z,-r) \approx \delta_n^{(r)} E_n(z,0)$ and

$$\langle f, g \cdot \delta_n^{(r)} E_n(z,0)\rangle \approx \langle f, g \cdot E_{k-n}(z,-r)\rangle \approx L(\ell_1 - 1 - r, f \times g).$$

- Conclusion: algebraicity of $\pi^{-\ell_1}\langle f, f\rangle^{-1} L(\ell_1 - 1 - r, f \times g)$
- Remark: algebraicity for (holomorphic projection of) $\langle f, g \cdot \delta_n^{(r)} E_n(z,0)\rangle$ can be derived from analysis of branching problem for holomoprhic discrete series of $\mathrm{SL}_2(\mathbb{R})$ embedded diagonally in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ (Harris)

Quaternionic Modular Forms

- Shimura's method has been expanded and generalized to many higher rank situations, often for L-functions associated to *holomorphic* modular forms, e.g. Siegel modular forms (Sp_{2n}) or holomorphic forms on unitary groups.
- There is a class of groups, the quaternionic groups, which do not necessarily have holomorphic discrete series but do have *quaternionic discrete series* (Gross-Wallach).
- Quaternionic modular forms (QMFs) are "special" automorphic forms on these groups associated to quaternionic discrete series, analogous to holomorphic modular forms for SL_2.
- QMFs have a good theory of Fourier expansion, with arithmetic properties (Gan-Gross-Savin, Pollack).

Adjoint L-Function

- Let G be the split exceptional group of type G_2. There is a subgroup $H \subseteq G$ isomorphic to $\mathrm{SU}(2,1)$. These groups have QMFs (Koseki-Oda, Hilado-McGlade-Yan).
- Hundley found an integral representation for the adjoint L-function of $\mathrm{SU}(2,1)$, which is amenable to QMFs.
- If φ is a cusp form on H of weight ℓ, and $E_\ell(g,s)$ is a certain degenerate Eisenstein series on G,

$$\langle \varphi, E_\ell(g,s) \rangle \approx L(s-1, \varphi, \mathrm{Ad}).$$

- We want to talk about algebraicity of our L-function at critical points (in the sense of Deligne), i.e. $L(k, \Pi, \mathrm{Ad})$ for $k = 2, 4, \ldots, \ell-2, \ell$.

Algebraicity Results

- Ingredient 1, Integral representation: ✓
- Ingredient 2, Control of Fourier coefficients / properties of Eisenstein series: When $s = \ell + 1$, then $E_\ell(g, s = \ell + 1)$ is a QMF.

Theorem (ongoing joint work with J. Johnson-Leung, F. McGlade, A. Pollack, M. Roy)

The degenerate quaternionic Eisenstein series on G_2 (and $B_n, D_n, F_4, E_6, E_7, E_8$) can be normalized to have algebraic Fourier coefficients.

- Cook these up: taking an eigenform $\varphi \in \Pi$

$$\frac{L(\ell, \Pi, \mathrm{Ad})}{\langle \varphi, \varphi \rangle} \in \pi^{\mathbb{Z}} \cdot \mathbb{Q}(\varphi).$$

- To get algebraicity results for critical values to the left, we need Ingredient 3, Differential Operators.

Algebraicity Results

Theorem (H.)

Let $n \geqslant 1$. For any integers $r \geqslant 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$, there exist differential operators $\mathcal{D}^n_{r,m}$ with the following properties:

- *If Φ is a QMF on G_2 of weight n, then $\varphi = \mathcal{D}^n_{r,m}\Phi|_{\mathrm{SU}(2,1)}$ is a QMF on $\mathrm{SU}(2,1)$ of weight $(n + \frac{r}{2}, m)$;*
- *the Fourier coefficients of φ are $\overline{\mathbb{Q}}$-linear combinations of the Fourier coefficients of Φ.*

Proof:

- Analysis of branching laws for quaternionic discrete series representations (H.Y. Loke) to find explicit recurrence relations for the operators.
- Compute effect of $\mathcal{D}^n_{r,m}$ on Fourier coefficients; related to invariant theory of binary cubic forms.

Thank you!